Analysis in Fractional Calculus and Asymptotics related to Zeta Functions

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Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared on this page and specified in the text.

It is not substantially the same as any that I have submitted, or that is being concurrently submitted, for a degree or diploma, or other qualification at the University of Cambridge or any other University or similar institution. I further state that no substantial part of my dissertation has already been submitted, or is being concurrently submitted, for any such degree, diploma, or other qualification at the University of Cambridge or any other University or similar institution.

It does not exceed the prescribed word limit for the relevant Degree Committee.

In Part 1, "Introduction", everything is review material, not my own research.
In Part 2, "Fractional calculus":

- §2.1 is entirely my own work. This research was published in the paper [46].
- §2.2 is the result of joint work with Dumitru Baleanu. This research was published in the paper [22].
- §2.3 is the result of joint work with Dumitru Baleanu and Thanasis Fokas. This research is currently under preparation for publication.
- §2.4 is the result of joint work with Dumitru Baleanu, but it comprises (insofar as individuals' contributions can be separated) only my contribution. This research was published in the papers [23] (§2.4.1–§2.4.6) and [48] (§2.4.7).
- §2.5 is the result of joint work with Dumitru Baleanu and, for §2.5.2, also with Hari Mohan Srivastava. This research comprises the papers [49] (§2.5.1) and [50] (§2.5.2), both currently under review for publication.
- §2.6 is entirely my own work. This research was published in the paper [47].

In Part 3, "Asymptotics for zeta functions":

- §3.1 is the result of joint work with Thanasis Fokas. Specifically, §3.1.1 is review material, §3.1.2 comprises his contribution, and §3.1.3–§3.1.6 comprise mine. This research was published in the paper [51].
- §3.2 comprises my contribution to a joint project with Thanasis Fokas and Euan Spence; it is entirely my own work apart from the review material in §3.2.1. This research was published in the paper [52].

Abstract

This thesis presents results in two apparently disparate mathematical fields which can both be examined – and even united – by means of pure analysis.

Fractional calculus is the study of differentiation and integration to non-integer orders. Dating back to Leibniz, this idea was considered by many great mathematical figures, and in recent decades it has been used to model many real-world systems and processes, but a full development of the mathematical theory remains incomplete.

Many techniques for partial differential equations (PDEs) can be extended to fractional PDEs too. Three chapters below cover my results in this area: establishing the elliptic regularity theorem, Malgrange–Ehrenpreis theorem, and unified transform method for fractional PDEs. Each one is analogous to a known result for classical PDEs, but the proof in the general fractional scenario requires new ideas and modifications.

Fractional derivatives and integrals are not uniquely defined: there are many different formulae, each of which has its own advantages and disadvantages. The most commonly used is the classical Riemann–Liouville model, but others may be preferred in different situations, and now new fractional models are being proposed and developed each year. This creates many opportunities for new research, since each time a model is proposed, its mathematical fundamentals need to be examined and developed.

Two chapters below investigate some of these new models. My results on the Atangana–Baleanu model proposed in 2016 have already had a noticeable impact on research in this area. Furthermore, this model and the results concerning it can be extended to more general fractional models which also have certain desirable properties of their own.

Fractional calculus and zeta functions have rarely been united in research, but one chapter below covers a new formula expressing the Lerch zeta function as a fractional derivative of an elementary function. This result could have many ramifications in both fields, which are yet to be explored fully.

Zeta functions are very important in analytic number theory: the Riemann zeta function relates to the distribution of the primes, and this field contains some of the most persistent open problems in mathematics. Since 2012, novel asymptotic techniques have been applied to derive new results on the growth of the Riemann zeta function.

One chapter below modifies some of these techniques to prove asymptotics to all orders for the Hurwitz zeta function. Many new ideas are required, but the end result is more elegant than the original one for Riemann zeta, because some of the new methodologies enable different parts of the argument to be presented in a more unified way.

Several related problems involve asymptotics arbitrarily near a stationary point. Ideally it should be possible to find uniform asymptotics which provide a smooth transition between the integration by parts and stationary phase methods. One chapter below solves this problem for a particular integral which arises in the analysis of zeta functions.

Acknowledgements

The four years spent on this PhD define a chapter in my life which has changed me completely, and more than once. The person I was at the beginning is hardly recognisable to me now as I approach the end. A PhD is a journey, and – despite the fact that it marks the beginning of an independent researcher – not one that can be undertaken alone. It would not be possible to thank everyone who has had an effect on me during these years, but this section represents an attempt to indicate my gratitude to those who helped my PhD progress in the most significant ways.

First of all, my supervisor Thanasis Fokas. He had faith in me to work well, produce good research, and successfully complete a PhD. I don't know how well I've lived up to his expectations, but I'm very grateful that he gave me the opportunity: not only to be a PhD student at all, but also to work with such an eminent mathematician on such exciting and important research topics.

Another professor to whom I owe a great deal, although he has no official role related to me, is Dumitru Baleanu. In the short time I've known him, he's taught me a huge amount both about the world of fractional calculus and also about academia more generally. Without him, I would have developed and learned far less during my PhD.

Three others in Cambridge who were there to advise me when I needed it are my academic advisor Anthony Ashton, my college tutor Yi Feng, and my friend Katarzyna Kowal. Their advice on topics ranging from giving presentations, to applying for research fellowships, to writing essays, research proposals, and theses, has been absolutely invaluable to me in navigating the PhD and the world of academia in general.

I must also express my gratitude to the Engineering and Physical Sciences Research Council for their research student grant, which enabled me to be here at all and which supported me through the first three years of my PhD.

Of course, the journey has been far from purely academic. I would like to thank my family, and in particular my mother, for her help and support in too many ways to count. Their unwavering moral support is something I appreciate far more than I talk about.

There have also been many friends who supported me, encouraged me, and helped me to see my path more clearly. From Katarzyna (again), to Dan, Jahn, and Jay (not forgetting the MD), to Canan and the friends from ICRAPAM 2017, to Maria Christina and the friends from Clare Hall ... although not all of these presences graced the full four years of my PhD, they have all at some time given me the companionship and support I needed, and helped to alter my outlook.

And last, but of course not least, God.

Contents

Part 1

Introduction

1.1 Fractional calculus

The differentiation and integration operators are among the most basic concepts in analysis. By applying them several times to the same function, we can compute nth derivatives and integrals for any $n \in \mathbb{N}$. By the Fundamental Theorem of Calculus, we can think of this as computing nth derivatives for any $n \in \mathbb{Z}$. But is it possible to go beyond the integers? Can we define the $\frac{1}{2}$th derivative of a function, or the $(-\pi)$th derivative? Even more ambitiously, can we venture into the complex plane and define the $(2+i)$th derivative?

The answer is yes. **Fractional calculus** – the study of differentiation to non-integer orders, so called because the first non-integer orders studied were rational – dates back to Leibniz himself (discussed informally in letters exchanged with l'Hôpital) and was also considered by mathematicians including Riemann [126], Abel [4], and Hardy and Littlewood [72, 73]. Detailed accounts of the history of the topic may be found in sources such as [105, 39]. But fractional calculus has only really begun to take off in the last forty years or so, with several specialised international conferences and journals now devoted to it, as well as many textbooks such as [127, 108, 105].

Note that when the order of differentiation and integration becomes a continuum, the difference between the two is often not clear-cut, and the term **differintegration** is used to cover both. When we do want to make a distinction, the difference between derivatives and integrals is usually defined in terms of the *real part* of the order of differintegration.

Indefinite integration of a function is only well-defined up to an additive constant, called a *constant of integration*. In the fractional context, such a constant must also be introduced for differentiation, since fractional derivatives are usually defined after, and in terms of, fractional integrals. We call this a *constant of differintegration*.

Fractional calculus has discovered a great many applications throughout many fields of science and engineering, and this has been a major contributor to its rise in popularity in recent decades.

- Some of these applications arise from the idea of **betweenness** which is so important to fractional calculus. For example, viscoelastic substances, which are in some sense 'between' viscous fluids and elastic solids, have properties which can be effectively modelled by fractional systems [19, 88, 98]. The fractionalisation concept can also be found in fractal geometry, whose key concept is the fractionalisation of dimension – constructing spaces with dimension 'between' the whole numbers – and this has applications in chaos theory.

- Another important property of fractional derivatives is that they are **nonlocal**: a function's fractional derivative at a particular point is not just influenced by the function's behaviour near that point. This novelty arising in fractional but not classical calculus has led to many applications in fields such as control theory and dynamical systems [117, 135].

Several different applications of fractional calculus are discussed in Hilfer's book [74] from 2000, but many more have been discovered since then. Further examples of areas where fractional calculus has been used in recent years include bioengineering [96], protein and tissue dynamics [68, 97], drug kinetics [38, 121, 130], epidemiology [15, 30], thermodynamics [141], nuclear dynamics [125], geohydrology [17], complexity theory [143], random walks [21, 147], image processing [116], and financial models [45].

An interesting feature of fractional calculus is that there are many different ways to define fractional derivatives and integrals, not all of which are equivalent to each other. This ambiguity gives rise to a rich tapestry of theories and techniques which each apply in different situations.

1.1.1 The Riemann–Liouville model

The most popular model of fractional calculus is given by the **Riemann–Liouville** formula, Definition 1.1.1. We shall examine this model in detail here and defer discussion of others to §1.1.2.

Definition 1.1.1 (Riemann–Liouville fractional differintegral)**.** Let x and ν be complex variables, and c be a constant in the extended complex plane (usually taken to be either 0 or $-\infty$). The fractional *integral* of a function $f(x)$ with respect to x, with constant of differintegration c, is defined by:

$$ {}_cD^{\nu}_x f(x) \coloneqq \tfrac{1}{\Gamma(-\nu)} \int_c^x (x-y)^{-\nu-1} f(y)\,\mathrm{d}y \quad , \quad \mathrm{Re}(\nu) < 0, \tag{1} $$

provided this expression is well-defined. The fractional *derivative* of $f(x)$ with respect to x, with constant of differintegration c, is defined by:

$$ {}_cD^{\nu}_x f(x) \coloneqq \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left({}_cD^{\nu-n}_x f(x) \right) \quad , \quad n \coloneqq \lfloor \mathrm{Re}(\nu) \rfloor + 1 \quad , \quad \mathrm{Re}(\nu) \geq 0, \tag{2} $$

provided this expression is well-defined.

Since x, ν, and c are defined in the complex plane, it is necessary to consider the issue of which path to integrate along from c to x and which branch to use for defining the function $(x-y)^{-\nu-1}$ for y on this path. Usually the straight line-segment contour $[c,x]$ is used, meaning that $\arg(x-y)$ is always equal to $\arg(x-c)$ independent of y. The choice of range for $\arg(x-c)$ usually depends on context, and the essential properties of Riemann–Liouville differintegrals remain unchanged whether we assume $\arg(x-c) \in [0,2\pi)$ or $\arg(x-c) \in (-\pi,\pi)$ or any other range. These issues are covered in [127, §22]. When all variables are real, as is often the case, most of these problems do not have to arise.

The constant of differintegration c tends to be fixed at either 0 or $-\infty$; other possibilities for c can usually be covered by the same arguments that work for these two cases. Note in particular that when $c = -\infty$, we can always take $\arg(x-c)$ to be 0, eliminating the problems of the previous paragraph.

The RL fractional integral (1) is a natural generalisation of Cauchy's formula for repeated integration [105, Chapter II]. The RL fractional derivative (2) is the extension of (1) by analytic continuation in ν; this follows from the fact that (1) gives $\frac{\mathrm{d}}{\mathrm{d}x}\big({}_cD_x^{\nu}f(x)\big) = {}_cD_x^{\nu+1}f(x)$ for $\mathrm{Re}(\nu) < -1$ already, so (2) will preserve analyticity as an extension of (1).

To see why both setting $c = 0$ and setting $c = -\infty$ can be sensible in different contexts, consider the following simple examples of RL differintegrals, both of which look exactly the way one would expect as natural generalisations of the classical derivative and integral expressions.

Lemma 1.1.2. *The Riemann–Liouville differintegral of a power function, with constant of differintegration $c = 0$, is given by*

$$ {}_0D_x^{\nu}(x^{\alpha}) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\nu+1)}\, x^{\alpha-\nu}, \tag{3}$$

for $\nu, \alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > -1$, where complex power functions are defined by the principal branch with arguments in the interval $(-\pi, \pi)$.

Proof. For fractional integrals ($\mathrm{Re}(\nu) < 0$), this follows from the definition (1) and the integral formula for the beta function. For fractional derivatives ($\mathrm{Re}(\nu) \leq 0$), we use standard differentiation on the result for fractional integrals. See [105] for more details. □

Lemma 1.1.3. *The Riemann–Liouville differintegral of an exponential function, with constant of differintegration $c = -\infty$, is given by*

$$ {}_{-\infty}D_x^{\nu}\left(e^{kx}\right) = k^{\nu}e^{kx}, \tag{4}$$

for $\nu, k \in \mathbb{C}$ with $k \notin \mathbb{R}_0^-$, where complex power functions are defined by the principal branch with arguments in the interval $(-\pi, \pi)$.

Proof (based on [127]). This follows from the definition of the gamma function, but care must be taken over the complex substitution in the integral. Note first that it will suffice to prove the result for $\mathrm{Re}(\nu) < 0$, since it will then follow for $\mathrm{Re}(\nu) \geq 0$ using the definition (2). Thus we assume $\mathrm{Re}(\nu) < 0$ and $k \notin \mathbb{R}_0^-$, and use the definition (1):

$$ {}_{-\infty}D_x^{\nu}\left(e^{kx}\right) = \tfrac{1}{\Gamma(-\nu)} \int_{-\infty}^{x} (x-y)^{-\nu-1} e^{ky}\,\mathrm{d}y. $$

Substituting $z = kx - ky$ yields

$$\begin{aligned}
{}_{-\infty}D_x^{\nu}\left(e^{kx}\right) &= \tfrac{1}{\Gamma(-\nu)}\int_{\infty}^{0}\left(\tfrac{z}{k}\right)^{-\nu-1}e^{kx-z}\left(\tfrac{1}{-k}\right)\mathrm{d}z \\
&= \tfrac{1}{\Gamma(-\nu)}e^{kx}\left(\tfrac{1}{k}\right)^{-\nu}\int_{0}^{\infty} z^{-\nu-1}e^{-z}\,\mathrm{d}z \\
&= e^{kx}\left(\tfrac{1}{k}\right)^{-\nu} = k^{\nu}e^{kx},
\end{aligned}$$

where for the last step we used the fact that k is not on the critical branch cut and therefore k and $\frac{1}{k}$ both have arguments in $(-\pi, \pi)$. □

Both of the above results are exactly what we would expect: they are the natural generalisations of the known formulae

$$\begin{gathered}
\tfrac{\mathrm{d}^k}{\mathrm{d}x^k}(x^{\alpha}) = (\alpha)(\alpha-1)(\alpha-2)\dots(\alpha-k+1)x^{\alpha-k}, \\
\tfrac{\mathrm{d}^k}{\mathrm{d}x^k}(e^{ax}) = a^k e^{ax},
\end{gathered}$$

from classical calculus (with $k \in \mathbb{N}$). Neither result remains valid if the $c = 0$ and $c = -\infty$ are swapped round, so neither of these possibilities for c can reasonably be scrapped. These equations also demonstrate one way in which Riemann–Liouville fractional calculus is a natural extension of classical calculus.

It is also possible to define fractional differintegrals with an upper limit of differintegration instead of a lower limit:

$$\begin{aligned}
{}_xD_b^{\nu}f(x) &= \frac{1}{\Gamma(-\nu)}\int_x^b (y-x)^{-\nu-1}f(y)\,\mathrm{d}y, && \mathrm{Re}(\nu) < 0; && (5) \\
{}_xD_b^{\nu}f(x) &= (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left({}_xD_b^{\nu-n}f(x)\right), n = \lfloor\mathrm{Re}(\nu)\rfloor + 1, && \mathrm{Re}(\nu) \geq 0. && (6)
\end{aligned}$$

These can be thought of as negated differintegrals: in the case when ν is a natural number, these fractional derivatives become powers of $-\frac{\mathrm{d}}{\mathrm{d}x}$ rather than powers of $\frac{\mathrm{d}}{\mathrm{d}x}$.

Another commonly seen justification for the Riemann–Liouville model is that it works together with Fourier and Laplace transforms in precisely the way one would expect. In other words, fractionally differintegrating a function corresponds to multiplying the transformed function by the corresponding power term. This result is codified by the following lemmas.

Lemma 1.1.4 (Fourier transforms of RL differintegrals)**.** *If $f(x)$ is a function with well-defined Fourier transform $\hat{f}(\lambda)$, and $\nu \in \mathbb{C}$ is such that ${}_{-\infty}D_x^{\nu}f(x)$ is well-defined, then the Fourier transform of ${}_{-\infty}D_x^{\nu}f(x)$ is $(-i\lambda)^{\nu}\hat{f}(\lambda)$.*

Proof (based on [127]). If $\mathrm{Re}(\nu) < 0$, then the definition (1) can be rewritten as a convolution: ${}_{-\infty}D_x^{\nu}f = f * \Phi$ where $\Phi(x) = \frac{x^{-\nu-1}}{\Gamma(-\nu)}$ when $x > 0$, $\Phi(x) = 0$ otherwise. Convolutions transform to products under the Fourier transform, so the result follows.

If $\mathrm{Re}(\nu) \geq 0$, the result follows from the fractional integral case (proved above) and the $\nu \in \mathbb{N}$ case (which is standard). □

Lemma 1.1.5 (Laplace transforms of RL integrals). *If $f(x)$ is a function with well-defined Laplace transform $\tilde{f}(\lambda)$, and $\nu \in \mathbb{C}$ with $\mathrm{Re}(\nu) < 0$ is such that ${}_0D_x^\nu f(x)$ is well-defined, then the Laplace transform of ${}_0D_x^\nu f(x)$ is $(-i\lambda)^\nu \tilde{f}(\lambda)$.*

Proof. As for Lemma 1.1.4. See also [105, Chapter III]. □

The corresponding result for Laplace transforms of fractional *derivatives* is more complicated, because of the initial value terms arising. It may be found in [105, Chapter IV].

Another important property to consider when we define fractional differintegrals is **composition**. Is the $\frac{1}{2}$th derivative of the $\frac{1}{2}$th derivative always equal to the 1st derivative? Surprisingly, the answer to questions like this is often 'no': it turns out that a semigroup property is not the most important feature to preserve when we extend calculus from integer orders to fractional orders. For the Riemann–Liouville model, the basic results on composition are summarised in the following two lemmas.

Lemma 1.1.6. *For any $x, \mu, \nu \in \mathbb{C}$ with $\mathrm{Re}(\mu) < 0$ and any function f continuous in a neighbourhood of c, the identity ${}_cD_x^\nu\left({}_cD_x^\mu f(x)\right) = {}_cD_x^{\mu+\nu} f(x)$ holds provided these differintegrals exist.*

Proof. This is a simple exercise in manipulation of double integrals, and may be found in [118, Chapter 2.3.2]. □

Lemma 1.1.7. *If $n \in \mathbb{N}$ and f is a C^n function such that one of ${}_cD^n\left({}_cD^\mu f(x)\right)$, ${}_cD_x^{n+\mu} f(x)$, ${}_cD_x^\mu\left({}_cD_x^n f(x)\right)$ exists, then all three exist and*

$$ {}_cD_x^n\left({}_cD_x^\mu f(x)\right) = {}_cD_x^{n+\mu} f(x) = {}_cD_x^\mu\left({}_cD_x^n f(x)\right) + \sum_{k=1}^{n} \frac{(x-c)^{-\mu-k}}{\Gamma(-\mu-k+1)} f^{(n-k)}(c). $$

Proof. The first identity follows directly from the definition (2) of Riemann–Liouville fractional derivatives. For the second, we can use induction on n, starting with the $\mathrm{Re}(\mu) < 0$ case and applying integration by parts, then proving the $\mathrm{Re}(\mu) \geq 0$ case by performing ordinary differentiation on the previous case. A more detailed proof can be found in [105, Chapter III]. □

Fractional calculus is essentially an extension of standard calculus, and thus a fundamental question is: **which properties are preserved?** For any theorem or method used in classical calculus, we can ask: does it still apply in a fractional scenario? Questions like this are what enables a solid platform to be built for the theoretical fundamentals of the field, after which the results obtained can be applied in many real-world situations.

Textbooks such as [105, 108, 127] cover many of the details of the basic theory of fractional calculus in standard models such as Riemann–Liouville. A nice example of results from classical calculus being successfully extended to RL fractional differintegrals can be found in the work of Osler from the early 1970s [111, 112, 113, 114, 115], in which he proved fractional versions of the product rule, chain rule, and Taylor's theorem. As we shall be using some of these results below, we state them here in the following three lemmas. (Note that the Cauchy formula for fractional differintegrals, Definition 1.1.11, is equivalent to the Riemann–Liouville one when both are defined; the reason for using it here is that it is more helpful given the complex context.)

Lemma 1.1.8 (RL fractional product rule)**.** *Let u and v be complex functions such that $u(x)$, $v(x)$, and $u(x)v(x)$ are all functions of the form $x^{\lambda}\eta(x)$ with $\mathrm{Re}(\lambda) > -1$ and η analytic on a domain $R \subset \mathbb{C}$. Then for any distinct $x, c \in R$ and any $\nu \in \mathbb{C}$, we have*

$$ {}_cD^{\nu}_x\big(u(x)v(x)\big) = \sum_{n=0}^{\infty} \tbinom{\nu}{n}\, {}_cD^{\nu-n}_x u(x)\, {}_cD^{n}_x v(x) \tag{7}$$

where all differintegrals are defined using the Cauchy formula.

Proof. See [113]. □

Lemma 1.1.9 (RL fractional chain rule)**.** *If y is a function of u and u is a smooth function of x such that y (as a function of x) is in the form $x^{\lambda}\eta(x)$ with $\mathrm{Re}(\lambda) > -1$ and η analytic on a domain $R \subset \mathbb{C}$, then*

$$ {}_cD^{\nu}_x y = \left(\tfrac{(x-c)^{-\nu}}{\Gamma(1-\nu)}\right) y + \sum_{n=1}^{\infty}\left(\tbinom{\nu}{n}\tfrac{(x-c)^{n-\nu}}{\Gamma(n-\nu+1)}\sum_{k=1}^{n}\left(\tfrac{\mathrm{d}^k y}{\mathrm{d}u^k}\sum_{P_1,\dots,P_n}\left(\prod_{i=1}^{n}\tfrac{i}{P_i!(i!)^{P_i}}\left(\tfrac{\mathrm{d}^i u}{\mathrm{d}x^i}\right)^{P_i}\right)\right)\right) \tag{8}$$

for any distinct $x, c \in R$ and any $\nu \in \mathbb{C}$, where all differintegrals are defined using the Cauchy formula and the summation over $(P_1, \dots, P_n)$ is over the set

$$\left\{(P_1,\dots,P_n) \in \left(\mathbb{Z}_0^+\right)^m : \sum_j P_j = k, \sum_j jP_j = n\right\}. \tag{9}$$

Proof. This follows from putting $u(x) = 1$ and $v(x) = y$ in the result of Lemma 1.1.8 and then using both Lemma 1.1.2 and the classical Faà di Bruno theorem. See also [112]. □

Lemma 1.1.10 (Fractional Taylor's theorem)**.** *If $f(x)$ is a complex function of the form $(x-c)^p g(x)$ where $c, p \in \mathbb{C}$, $\mathrm{Re}(p) > -1$, and g is analytic on a ball $R \coloneqq B_r(c)$, and $z_0 \in R$ is such that the set $C \coloneqq \{x \in \mathbb{C} : |x - z_0| = |c - z_0|\}$ is contained in R, then*

$$ {}_cD^{\nu}_x f(x) = \int_{-\infty}^{\infty} \tfrac{(x-z_0)^{\omega+\alpha}}{\Gamma(\omega+\alpha+1)}\, {}_cD^{\omega+\alpha+\nu}_x f(x)\,\mathrm{d}\omega$$

for all $x \in C$, $\alpha, \nu \in \mathbb{C}$ such that $\mathrm{Re}(p-\nu) > -1$, *where all differintegrals are defined using the Cauchy formula.*

Proof. See [115]. $\square$

But despite the solid existing foundation for the theory of RL fractional calculus, there is always more to be done. For any result in classical calculus, it is possible to ask whether the same result is valid in fractional calculus.

This is especially important in the theory of PDEs: fractional PDEs are a growing field of interest, with entire textbooks written about them and their applications [87, 118]. A huge variety of methods have been devised for solving them, including by extending known results of classical calculus: see for example [119, 146, 25] among many others. However, fractional PDEs remain much less well understood than standard PDEs.

I have been working in this field for several years and achieved the following results:

- A *fractional elliptic regularity theorem*, concerning the regularity behaviour of the solutions to certain linear fractional partial differential equations.
- A *fractional Malgrange–Ehrenpreis theorem*, establishing the existence of fundamental solutions to certain fractional partial differential equations.
- A *fractional unified transform method*, used to construct explicit solutions to a certain class of fractional partial differential equations.

The first three chapters below are devoted to these results. Specifically: in §2.1 the fractional elliptic regularity theorem is formulated and proved using a bootstrapping proof; in §2.2 the fractional Malgrange–Ehrenpreis theorem is formulated and proved in two different ways; and in §2.3 the fractional unified transform method is developed and tested. In all three cases, the new work is done by using some methods adapted from the classical scenario, with the addition of extra techniques and ideas required for the harder, more general, fractional case.

1.1.2 Other models old and new

As already mentioned above, there are many alternatives to the Riemann–Liouville model of fractional calculus, and here we introduce some of the most popular and widely applicable. Some of them are only slight variations of RL, or equivalent to it on certain function spaces; others are much further removed from it and have very different properties. It is important to be aware of these different ways of defining fractional differintegrals, because they each provide different possibilities and properties, and one may be more useful than another depending on the specific scenario in question.

The **Cauchy** formula for differintegration, Definition 1.1.11, is formed by replacing the straight integral in (1) by a complex contour integral. This definition is equivalent to the Riemann–Liouville one wherever both are defined [108, Chapter 3], but it can be more useful for applications in complex analysis, being the natural generalisation of Cauchy's integral formula from complex analysis. Note that unlike the RL integral formula (1), this formula works for all ν except negative integers, which are covered by standard integration.

Definition 1.1.11 (**Cauchy fractional differintegral**)**.** Let x and ν be complex variables, and c be a constant in the extended complex plane. For $\nu \in \mathbb{C}\backslash\mathbb{Z}^-$, the νth derivative of a function f is

$$ {}^{\mathbb{C}}_{c}D^{\nu}_{x}f(x) \coloneqq \frac{\Gamma(\nu+1)}{2\pi i}\int_{\mathcal{H}}(y-x)^{-\nu-1}f(y)\,\mathrm{d}y, \tag{10}$$

where $\mathcal{H}$ is a finite Hankel-type contour with both ends at c and circling once counterclockwise around x. This is well-defined provided f is analytic in a neighbourhood of the line segment $[c, x]$ and, if c is infinite, provided f has sufficient decay properties at c.

The **Caputo** formula for fractional derivatives, Definition 1.1.12, is motivated by looking at the definition of Riemann–Liouville derivatives (2) and posing the question: why use ordinary derivatives of fractional integrals rather than fractional integrals of ordinary derivatives? Swapping the operations gives an alternative definition of fractional derivatives, which is often more useful in applications involving initial value problems:

Definition 1.1.12 (**Caputo fractional derivative**)**.** Let x and ν be complex variables, and c be a constant in the extended complex plane. For $\mathrm{Re}(\nu) \geq 0$, the νth derivative of a function f is

$$ {}^{C}_{c}D^{\nu}_{x}f(x) \coloneqq {}_{c}D^{\nu-n}_{x}\left(\tfrac{d^n f}{dx^n}\right),\ n \coloneqq \lfloor \mathrm{Re}(\nu)\rfloor + 1, \tag{11}$$

where the fractional integral operator ${}_{c}D^{\nu-n}_{x}$ is defined by (1), provided this expression is well-defined.

Note that fractional *integrals* in the Caputo context are exactly Riemann–Liouville integrals; a new definition is not needed for them. We can also see from Lemma 1.1.7 above that the RL and Caputo derivatives are not equivalent in general, but that there is a simple relationship between them in terms of the values of the function f and its derivatives at c. This allows us to derive conditions for when they *are* equal: for example, if $c = -\infty$, such conditions would amount to decay properties on f and its derivatives.

The **Grünwald–Letnikov** formula for fractional differintegrals, Definition 1.1.13, is defined by the limit of a series. It can be seen [108] as a natural extension of both the formula $\frac{f(x+h)-f(x)}{h}$ for standard derivatives and the Riemann integration formula for standard integrals. The advantage of the Grünwald–Letnikov formula over those above is that it is easier to compute numerically to a given order of approximation: it is expressed as a limit as $N \to \infty$, so a good approximation can be found by taking some large value of N. The disadvantage is that it is much uglier than the other formulae and almost impossible to determine analytically for most given functions f. However, it is equivalent to the Riemann–Liouville definition wherever both are defined [108].

Definition 1.1.13 (**Grünwald–Letnikov fractional differintegral**). Let x and ν be complex variables, and c be a constant in the extended complex plane. The νth derivative of a function $f(x)$ with respect to x, with constant of differintegration c, is

$$ {}^{GL}_{c}D^{\nu}_{x}f(x) \coloneqq \lim_{N\to\infty} \Big(\frac{1}{\Gamma(-\nu)} \big(\frac{x-c}{N} \big)^{-\nu} \sum_{k=0}^{N-1} \frac{\Gamma(k-\nu)}{\Gamma(k+1)} f\big(x - k(\tfrac{x-c}{N})\big) \Big), \tag{12} $$

provided that this limit exists.

More recently, several models of fractional calculus have been introduced in which the power function kernel that appears in the Riemann–Liouville definition (1) is replaced by a different function. The underlying motivation is to investigate whether it is possible to construct other types of fractional operators which have nonsingular kernel and which can better describe the dynamics of certain nonlocal phenomena. This idea inspired Caputo and Fabrizio [29] in 2015 to propose what is now called the **Caputo–Fabrizio** formula, in which the power function kernel is replaced by an exponential function.

Definition 1.1.14 (**Caputo–Fabrizio fractional derivative**). Let x and ν be real variables, and c be a constant on the extended real axis. For $0 < \nu < 1$, the νth derivative of a function f is

$$ {}^{CF}_{c}D^{\nu}_{x}f(x) \coloneqq \frac{M(\nu)}{1-\nu} \int_{c}^{x} \exp\left(\frac{-\nu}{1-\nu}(x-y) \right) f'(y)\,\mathrm{d}y, \tag{13} $$

provided this expression is well-defined, where M is a multiplier function (introduced to allow the possibility of weighting one order of differentiation more than another) which satisfies $M(0) = M(1) = 1$.

The Caputo–Fabrizio definition has already found applications in areas such as diffusion modelling [78] and mass-spring-damper systems [9].

Other definitions along similar lines involve using the Mittag-Leffler function as a kernel. This function is known to be very significant in fractional calculus [99, 101, 138, 131], and its properties have been exhaustively studied in this connection [70, 75]. Fractional models with Mittag-Leffler kernel include the **Atangana–Baleanu** [18] and **Prabhakar** [122] models. These are discussed in detail in later chapters, and thus their precise definitions are deferred to there: specifically, the Atangana–Baleanu definition is given in §2.4.2 and the Prabhakar definition in §2.5.2. In the case of the Atangana–Baleanu model, a formal rigorous definition including the required function space for f is one of the achievements represented by this thesis.

After being introduced to such novel fractional models via the Atangana–Baleanu (AB) one, I have been working in this subfield of fractional calculus with some success, achieving a number of results which are detailed below. Specifically: in §2.4 the AB model is explored in depth, from constructing new formulae for AB derivatives to solving various families of fractional ODEs to proving or disproving important fundamental properties such as the semigroup property, product rule, and Taylor's theorem in this framework; and in §2.5 some of the same ideas are taken further, used to construct a new fractional model based on the AB one, and also to prove new properties of the existing Prabhakar model of fractional calculus.

1.1.3 Fractional calculus and zeta functions

This part of the thesis is difficult to classify, falling as it does neatly between the two main topics of fractional calculus and analytic number theory. I have opted to categorise it under fractional calculus, because it does not relate specifically to the *asymptotics* of zeta functions. However, for the definitions of the Riemann zeta function and its generalisations, the reader is referred to §1.2, specifically equations (15)–(17).

Despite the increasing usefulness and applications of fractional calculus, it has so far been largely neglected as a tool in analytic number theory. I believe that this is a missed opportunity and a ripe field for new research. For example, since fractional derivatives are nonlocal operators, they may be a useful tool for capturing the global properties of zeta functions which have been so elusive.

The first attempt to apply fractional calculus to zeta functions appears to have been by Keiper in his master's thesis [85], in which he established the following formula for the Riemann zeta function $\zeta(\sigma+it)$ as a fractional differintegral:

$$\zeta(s)=\sum_{k=1}^{m-1}k^{-s}+\frac{(-1)^s}{\Gamma(s)}\lim_{|c|\to\infty}{}_cD_z^{s-1}\left[\frac{\Gamma'(z)}{\Gamma(z)}\right]_{z=m},\qquad \mathrm{Re}(s)>1,m\in\mathbb{N}. \tag{14}$$

This is very conceptually interesting, in that it formed a new connection between two different branches of mathematics, but its usefulness is limited since it is only valid for $\sigma>1$, the region in which there is already a convergent series for the zeta function.

More recently, Guariglia [71] has applied a different model of fractional calculus, known as the Ortiguiera–Caputo model, to the Riemann zeta function. His expression for the fractional derivative of $\zeta(s)$ was a convergent infinite series of powers of logarithms. Furthermore, Lin and Srivastava [93] investigated generalisations of the Riemann zeta function, to the Lerch zeta function and beyond, and found fractional relations between some of these generalised functions. In particular, they showed that the generalised Hurwitz-Lerch zeta function defined by

$$\Phi_{\mu,\nu}^{(\rho,\sigma)} := \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}} \frac{z^n}{(n+a)^s}$$

can be expressed as a fractional differintegral of the standard Lerch zeta function (17).

I have shed new light on the connection between fractional calculus and analytic number theory, by expressing the Lerch zeta function $L(\lambda, x, s)$ as a fractional differintegral of an elementary function, in an identity which can be proven valid for all s by a complex argument involving analytic continuation. One chapter below, §2.6, is devoted to the derivation and analysis of this result.

1.2 Asymptotics for zeta functions

Zeta functions are among the most important objects in the challenging field of analytic number theory. Although easy to define, they turn out to be very hard to analyse, and even some of their most fundamental properties remain unknown after over a century of study. Some of the most persistent open problems in mathematics, such as the famous Riemann Hypothesis, concern these functions.

The most famous zeta function is the **Riemann zeta function**, an analytic function defined on a right half plane by

$$\zeta(s) \coloneqq \sum_{n=1}^{\infty} n^{-s}, \quad s = \sigma + it, \sigma > 1, \tag{15}$$

and defined for all $s \in \mathbb{C}\backslash\{1\}$ by analytic continuation [137]. This function in particular has been the subject of intense study for nearly two hundred years, mostly due to its connection with the distribution of prime numbers [40, 79, 137]. The Riemann Hypothesis states that all zeros of the Riemann zeta function are either of the form $-2n, n \in \mathbb{N}$, or have real part $\frac{1}{2}$. The weaker Lindelöf Hypothesis states that the Riemann zeta function on the critical line, $\zeta(\frac{1}{2}+it)$, grows more slowly than any power function t^{ϵ}, $\epsilon > 0$, as t tends to infinity. Both of these statements remain open problems; however, new progress has been made very recently towards the Lindelöf Hypothesis, as detailed below.

The Riemann zeta function can be generalised in a number of directions: for example, the Dirichlet L-functions are number-theoretical generalisations depending on both the complex variable s and also a Dirichlet character modulo some base d, while the Hurwitz zeta function and Lerch zeta function are analytic generalisations depending on two or three independent complex variables.

Specifically, the **Hurwitz zeta function** is defined on a right half plane by

$$\zeta(x,s) \coloneqq \sum_{n=0}^{\infty} (n+x)^{-s}\ , \ \ \mathrm{Re}(x) > 0, s = \sigma + it, \sigma > 1, t \in \mathbb{R}, \tag{16}$$

and defined for all $s \in \mathbb{C}\backslash\{1\}$ by analytic continuation [12, Chapter 12]. The existence of the additional parameter x leads to interesting results which do not have analogues for the Riemann zeta function; see for example [142], [20], [10], [84], [102], and p. 73 in [35]. And the **Lerch zeta function** is defined on a right half plane by

$$L(\lambda,x,s) \coloneqq \sum_{n=0}^{\infty} (n+x)^{-s} e^{2\pi i\lambda n}\ , \ \ \mathrm{Re}(s) > 1, \mathrm{Re}(x) > 0, \mathrm{Im}(\lambda) \geq 0, \tag{17}$$

and by analytic continuation for (λ, x, s) in larger domains [90], extending to a universal cover of the manifold $\mathbb{C}\backslash\mathbb{Z} \times \mathbb{C}\backslash\mathbb{Z}_0^- \times \mathbb{C}$.

It is clear that the Riemann, Hurwitz, and Lerch zeta functions are related by the following identities:

$$\zeta(s) = \zeta(1, s); \quad \zeta(x, s) = L(0, x, s).$$

Many of the techniques used for analysing the Riemann zeta function and Dirichlet L-functions, such as the Euler product formula, have no general analogues for the Hurwitz or Lerch zeta functions. This is because the latter functions have a less direct connection to number theory, and are more readily studied using analytic methods. However, many important facts about the Riemann zeta function do have analogues in the Hurwitz and Lerch cases [123, 64, 66], which are even proved in some cases by analogous methods. Furthermore, analysing the Hurwitz and Lerch zeta functions can still be significant for number theory, because they include the Riemann zeta function as a special case.

In the last few years, a new approach has been proposed by Fokas for analysing zeta functions. The ideas involved are purely analytic, with the number-theoretical aspects of the zeta function taking a back seat. Techniques from complex and applied analysis, such as Plemelj formulae and asymptotics for exponential integrals, are brought to bear on the Riemann zeta function and the various integrals and series associated with it, in order to yield new asymptotic results for the zeta function itself. This work, spread across several manuscripts by Fokas and others [57, 56, 83], represents major new progress towards the Lindelöf Hypothesis.

The following classical asymptotic formula for $\zeta(s)$, proved in e.g. [137, Theorem 4.15], is known as the **approximate functional equation**:

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \chi(s) \sum_{n \leq y} \frac{1}{n^{1-s}} + O\Big(x^{-\sigma} + |t|^{\frac{1}{2}-\sigma} y^{\sigma-1}\Big), \tag{18}$$

where

$$xy = \tfrac{t}{2\pi}, 0 < \sigma < 1, t \to \infty,$$

and the entire function $\chi(s)$ is defined by

$$\chi(s) \coloneqq \tfrac{(2\pi)^s}{\pi} \Gamma(1-s) \sin\left(\tfrac{\pi s}{2}\right), \ s \in \mathbb{C}. \tag{19}$$

Siegel, in his classical paper [129] following Riemann's unpublished notes, found expressions for the error term in (18) to all orders for the important particular case $x = y = \sqrt{\frac{t}{2\pi}}$.

The recent work of Fokas and Lenells [57] presents formulae analogous to those of Siegel which are valid for any x, y satisfying $xy = \frac{t}{2\pi}$. The starting point of this analysis was the following exact formula, proved in Theorem 2.1 of [57]:

$$\zeta(s) = \chi(s)\Bigg[\sum_{n=1}^{\lfloor \eta/2\pi \rfloor} n^{s-1} \\ + \frac{1}{(2\pi)^s}\left(-\frac{\eta^s}{s} + e^{i\pi s/2}\int_{-i\eta}^{\infty e^{i\phi_1}} \frac{z^{s-1}}{e^z-1}\,\mathrm{d}z + e^{-i\pi s/2}\int_{i\eta}^{\infty e^{i\phi_2}} \frac{z^{s-1}}{e^z-1}\,\mathrm{d}z \right)\Bigg], \quad (20)$$

valid for

$$0 < \eta < \infty, -\tfrac{\pi}{2} < \phi_1, \phi_2 < \tfrac{\pi}{2}, s \in \mathbb{C}.$$

The exact identity (20) was then used, together with the asymptotic techniques of integration by parts and steepest descent, as well as an approach following that of Siegel [129], to derive asymptotic expressions for the Riemann zeta function $\zeta(\sigma + it)$ which are valid to all orders in t. Different methods were used according to the range of values for the new parameter η, giving rise to several different asymptotic expressions valid for different ranges of η: the details may be found in [57].

Still more recently, Fokas has provided [56] the formal proof that a certain double sum, which may be considered as a variant of $|\zeta(\frac{1}{2} + it)|^2$, satisfies the relevant analogue of the Lindelöf Hypothesis. The starting point of this analysis is the following exact formula:

$$\frac{t}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \mathrm{Re}\left[\frac{\Gamma(it - iut)\Gamma(\sigma + iut)}{\Gamma(\sigma + it)}\right] |\zeta(\sigma + iut)|^2\,\mathrm{d}u = -\mathcal{G}(\sigma, t), \qquad \sigma \in [0,1], t \in \mathbb{R}^+, \quad (21)$$

where $\mathcal{P}$ denotes a principal value integral (here taken with respect to $u = 1$) and $\mathcal{G}$ is defined by

$$\mathcal{G}(\sigma, t) = \begin{cases} \zeta(2\sigma) + \left(\frac{\Gamma(1-\sigma+it)}{\Gamma(\sigma+it)} + \frac{\Gamma(1-\sigma-it)}{\Gamma(\sigma-it)}\right)\Gamma(2\sigma-1)\zeta(2\sigma-1) + \frac{2(\sigma-1)\zeta(2\sigma-1)}{(\sigma-1)^2+t^2}, & \sigma \neq \frac{1}{2}; \\ \mathrm{Re}\left[\frac{\Gamma'(\frac{1}{2}+it)}{\Gamma(\frac{1}{2}+it)}\right] + 2\gamma - \log(2\pi) + \frac{2}{1+4t^2}, & \sigma = \frac{1}{2}. \end{cases}$$

The exact formula (21) was proved in [56] using an adaptation of the Plemelj formulae method for solving Riemann-Hilbert problems. The infinite interval of integration in (21) was then split into many shorter intervals, in order that the asymptotics of each shorter integral could be computed individually.

Many of the resulting asymptotic formulae are expressed in terms of double sums, e.g. series of the form

$$\sum_{m_1, m_2}\sum \frac{1}{m_1^s (m_1 + m_2)^{\bar{s}}},$$

and other more complicated expressions in the same vein. These in turn can be analysed using existing methods from analytic number theory [83], and explicit asymptotics may be computed for these series.

The final result of Fokas's work, emerging from a detailed analysis of the different sections of the integral in (21) and the many subproblems involving asymptotics of various different integrals and series, is the following statement: the sum of $|\zeta(\frac{1}{2}+it)|^2$ with a certain double series, which depends explicitly on the parameter $\epsilon > 0$, is of order $O(t^\epsilon)$ as $t \to \infty$. Taking into consideration that ϵ is arbitrary, this equation suggests the validity of the Lindelöf Hypothesis.

Several researchers have played a role in this ambitious project, and I have contributed in two different ways. My results in this area are covered in the final chapters of the thesis.

Firstly, in §3.1 I have extended the analysis of [57] on asymptotics to all orders of the Riemann zeta function, computing asymptotics to all orders of the Hurwitz zeta function. This involved a lot of extra work compared to the simpler Riemann zeta case, since the approach of Siegel is no longer valid for the Hurwitz zeta function. However, this apparent setback actually turned out to be an advantage, because it inspired the construction of a unified analysis, where the same method for asymptotics works regardless of the value of the parameter η.

Secondly, in §3.2 I have provided the rigorous analysis for one of the subproblems arising in the work of [56]: specifically, the asymptotics of one of the integrals obtained from (21) by splitting the interval of integration into various subintervals. The starting point for this analysis is a particular exponential integral, for which I have proved a uniform asymptotic formula, equally valid regardless of how close a stationary point is to the interval of integration. Although only first-order asymptotics were required for the work of [56], I derived the asymptotics to all orders. This method may have wider applicability to the general problem of finding uniform asymptotics in the neighbourhood of a stationary point.

Part 2

Fractional calculus

This Part of the manuscript contains my research on fractional calculus, as indicated briefly in §1.1 of the introduction. A more detailed summary is as follows.

- §2.1 concerns the elliptic regularity theorem for fractional PDEs. The main result is Theorem 2.1.12. The subchapters are §2.1.1 to set up the problem, §2.1.2 to provide a detailed proof of the main result, and §2.1.3 to discuss example applications and potential extensions.

- §2.2 concerns the Malgrange–Ehrenpreis theorem for fractional PDEs. The main results are Theorem 2.2.2 and Theorem 2.2.4. The subchapters are §2.2.1 to set up the problem, §2.2.2 and §2.2.3 to provide detailed proofs of Theorems 2.2.2 and 2.2.4 respectively, and §2.2.4 to discuss example applications and potential extensions.

- §2.3 concerns the unified transform method for fractional PDEs. The main result is Theorem 2.3.3. The subchapters are §2.3.1 to demonstrate the method for classical PDEs, §2.3.2 to set up the problem for fractional PDEs, §2.3.3–§2.3.5 to work through the steps of the method as it applies to fractional PDEs, §2.3.6 to summarise and verify the results achieved, and §2.3.7 to work through an example in detail and discuss potential extensions.

- §2.4 concerns an analysis of Atangana–Baleanu fractional calculus. It covers many aspects of this fractional model, so there is no single main result. The subchapters are §2.4.1 to introduce the model, §2.4.2 to establish rigorous definitions and also a series formula which has many consequences, §2.4.3 to solve various linear fractional ODEs in the AB model, §2.4.4 to solve various nonlinear fractional ODEs in the AB model, §2.4.5 to examine the semigroup property in this model, §2.4.6 to prove analogues of the product rule and chain rule for AB derivatives, and §2.4.7 to prove analogues of the mean value theorem and Taylor's theorem for AB derivatives.

- §2.5 concerns extending the ideas from the previous chapter to more general models of fractional calculus. Again many different aspects are covered and there is no single main result. The subchapters are §2.5.1 to introduce and analyse a new definition of fractional calculus based on iterations of the AB formula, and 2.5.2 to examine the generalised Prabhakar model and prove many results analogous to those now established for the AB model, including a series formula, a product rule, and a chain rule.

- §2.6 concerns a new expression for the Lerch zeta function as a fractional derivative. The main result is Theorem 2.6.2. The subchapters are §2.6.1 to introduce the topic, §2.6.2 to provide a detailed proof of the main result and its immediate corollary for the Riemann zeta function, and §2.6.3 to analyse further these results and their ramifications.

2.1 The elliptic regularity theorem

2.1.1 Introduction and setup

The **elliptic regularity theorem** is an important result in the theory of partial differential equations. In its most general form, it says that for any PDE satisfying certain conditions, there are regularity properties of the solution function which depend naturally on the regularity properties of the forcing function. This is useful in cases where the solution function cannot be constructed explicitly: more information about its essential properties is the next best thing to an analytic solution.

Here we shall focus on the version of the elliptic regularity theorem given in Theorem 2.1.5, in which the PDE must be linear and elliptic with constant coefficients, and 'regularity' is defined in terms of Sobolev spaces. Before stating the theorem, we fix our notation for the spaces of functions and distributions which we will use below.

Definition 2.1.1. Let X be a domain in $\mathbb{R}^n$. The space $\mathcal{D}(X)$ of **test functions** on X is defined to be the set of smooth compactly supported functions $X \to \mathbb{R}$, with the topology defined by saying that a sequence (ϕ_m) converges to 0 in $\mathcal{D}(X)$ if and only if there exists a compact set $K \subset X$ such that all ϕ_m have supports contained in K and the partial derivative $\partial^\alpha \phi_m$ tends uniformly to 0 for all multi-indices α.

The space $\mathcal{D}'(X)$ of **distributions** on X is defined to be the set of all linear maps $u : \mathcal{D}(X) \to \mathbb{C}$ such that for any compact set $K \subset X$, there exist constants $C \in \mathbb{R}^+$ and $N \in \mathbb{Z}_0^+$ such that for any $\phi \in \mathcal{D}(X)$ with support contained in K,

$$\Big|\langle u, \phi \rangle\Big| \leq C \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \phi| \,.$$

Definition 2.1.2. Let X be a domain in $\mathbb{R}^n$. The space $\mathcal{E}(X)$ is defined to be the set of smooth functions $X \to \mathbb{R}$, with the topology defined by saying that a sequence (ϕ_m) converges to 0 in $\mathcal{E}(X)$ if and only if for all compact sets $K \subset X$, the partial derivative $\partial^\alpha \phi_m$ tends to 0 uniformly on K for all multi-indices α.

The space $\mathcal{E}'(X)$ of **distributions of compact support** on X is defined to be the set of all linear maps $u : \mathcal{E}(X) \to \mathbb{C}$ such that there exist a compact set $K \subset X$ and constants $C \in \mathbb{R}^+$, $N \in \mathbb{Z}_0^+$ such that for any $\phi \in \mathcal{E}(X)$,

$$\Big|\langle u, \phi \rangle\Big| \leq C \sum_{|\alpha| \leq N} \sup_K |\partial^\alpha \phi| \,.$$

Definition 2.1.3. The space $\mathcal{S}(\mathbb{R}^n)$ of **Schwartz functions** is defined to be the set of smooth functions $\phi : \mathbb{R}^n \to \mathbb{C}$ such that

$$\|\phi\|_{\alpha,\beta} \coloneqq \sup_{x \in \mathbb{R}^n} \left| x^\alpha \partial^\beta \phi(x) \right|$$

is finite for all multi-indices α, β, with the topology defined by saying that a sequence (ϕ_m) converges to 0 in $\mathcal{S}(X)$ if and only if $\|\phi\|_{\alpha,\beta} \to 0$ for all α, β.

The space $\mathcal{S}'(\mathbb{R}^n)$ of **tempered distributions** is defined to be the set of all linear maps $u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ such that there exist constants $C \in \mathbb{R}^+$ and $N \in \mathbb{Z}_0^+$ such that for any $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\left| \langle u, \phi \rangle \right| \leq C \sum_{|\alpha|,|\beta| \leq N} \|\phi\|_{\alpha,\beta}.$$

Definition 2.1.4. For any real number s, the sth **Sobolev space** on $\mathbb{R}^n$ is

$$H^s(\mathbb{R}^n) := \{u \in \mathcal{S}'(\mathbb{R}^n) : \hat{u} \in L^2_{loc}(\mathbb{R}^n), ||u||_{H^s} < \infty\},$$

where the Sobolev norm $|| \cdot ||_{H^s}$ is defined by

$$||u||_{H^s} := \left(\int_{\mathbb{R}^n} |\hat{u}(\lambda)|^2 \left(1 + |\lambda|^2\right)^s \, \mathrm{d}\lambda \right)^{1/2}.$$

For a general domain $X \subset \mathbb{R}^n$, the sth Sobolev space on X is defined to be

$$H^s_{loc}(X) := \{u \in \mathcal{D}'(X) : u\phi \in H^s(\mathbb{R}^n) \text{ for all } \phi \in \mathcal{D}(X)\}.$$

Theorem 2.1.5 (Elliptic regularity theorem)**.** *Let $P(D)$ be an elliptic partial differential operator given by a complex n-variable Nth-order polynomial P applied to the differential operator $D := -i\frac{\partial}{\partial x}$ where x is a variable in $\mathbb{R}^n$. If X is a domain in $\mathbb{R}^n$ and $u, f \in \mathcal{D}'(X)$ satisfy $P(D)u = f$, then*

$$f \in H^s_{loc}(X) \Rightarrow u \in H^{s+N}_{loc}(X).$$

Proof. See [59, Chapter 9]. □

Related, more general, results are already known from the theory of pseudodifferential operators; see e.g. [5, Theorem 7.13] for an example of an elliptic regularity theorem in this setting. However, it is not necessary to introduce the full machinery of pseudodifferential operators – with associated stronger conditions on the forcing and solution functions – in order to obtain a useful analogue of Theorem 2.1.5 for fractional differential equations.

Elliptic fractional PDEs have already been studied in papers such as [26, 33, 28, 36], which present various methods for analysing the solutions of certain classes of elliptic fractional PDE. The work of this chapter fits in with such results by providing a quick way of establishing important regularity properties of linear elliptic fractional PDEs. The bootstrapping proof used in [59] to prove the classical Theorem 2.1.5 can be adapted to prove an elegant analogous result which is valid for a large class of Riemann–Liouville fractional PDEs. Naturally, some modifications and extra lemmas are required to deal with the new operators involved in the fractional problem; the proofs of Lemmas 2.1.10

and 2.1.11 provided the most new challenges. Nevertheless, the final result of Theorem 2.1.12 should be as significant for the study of fractional PDEs as the original elliptic regularity theorem was for the study of classical PDEs.

We start by defining our notation clearly. Let $x \in \mathbb{R}^n$ be an n-dimensional variable, and let D denote the modified n-dimensional differential operator $-i\,{}_{-\infty}D_x$. In other words, the differential operator D^α is defined by

$$D^\alpha f(x) = e^{-i\pi\alpha/2}\,{}_{-\infty}D_x^\alpha f(x),$$

where the vector differential operator with respect to x is of Riemann–Liouville type. We use the constant of differintegration $c = -\infty$ so that we can make use of Fourier transforms in the proof (by Lemma 1.1.4), and also so that the Riemann–Liouville and Caputo fractional derivatives of certain functions are equal (by the remark following Definition 1.1.12), which is required at a certain stage in the proof.

Let P be a finite linear combination of power functions, i.e.

$$P(\lambda) = \sum_\alpha c_\alpha \lambda^\alpha,$$

where α is a fractional multi-index in $(\mathbb{R}_0^+)^n$ and the sum is finite. This defines a fractional differential operator $P(D)$, with D defined as above, and the fractional partial differential equation we shall be considering is of the form

$$P(D)u = f.$$

When discussing fractional differintegrals of distributions, we use the natural definition provided in [144, Chapter 4]. In contrast to standard derivatives of distributions, this definition requires the distributions to be acting on the Lizorkin space

$$\Phi(X) = \{\phi \in \mathcal{D}(X) : \phi^{(n)}(0) = 0\ \forall n \in \mathbb{Z}_0^+\},$$

which was introduced in [95]. Thus, in the results below, we shall take the solution space for u to be the space $\Phi'(X)$, which is defined from the Lizorkin space in the same way as $\mathcal{D}'(X)$ was constructed in Definition 2.1.1.

Definition 2.1.6. The **order** ν of the operator $P(D)$ defined above is the maximal $|\alpha|$ such that $c_\alpha \neq 0$. Note that ν is not necessarily an integer, and that since P is a finite sum, there exists $\epsilon > 0$ such that $|\alpha| \leq \nu - \epsilon$ for every α such that $c_\alpha \neq 0$ and $|\alpha| < \nu$.

Definition 2.1.7. The **principal symbol** of $P(D)$ is defined to be the function $\sigma_P(\lambda) = \sum_{|\alpha|=\nu} c_\alpha \lambda^\alpha$. The operator $P(D)$ is said to be **elliptic** if $\sigma_P(\lambda) \neq 0$ for all nonzero $\lambda \in \mathbb{R}^n$.

2.1.2 Derivation of the result

Before proving the main result Theorem 2.1.12, we need to establish several lemmas which will be used in the proof.

Lemma 2.1.8. *If $P(D)$ is a νth-order elliptic fractional partial differential operator as above, then there exist positive real constants C, R such that for any $\lambda \in \mathbb{C}^n$ with $|\lambda| > R$, the function P satisfies $|P(\lambda)| \geq C(1+|\lambda|^2)^{\nu/2}$.*

Proof. First consider the non-fractional case, i.e. where P is a polynomial. Here $|\sigma_P|$ is a continuous positive function on the compact domain $|\lambda| = 1$, so it has a positive lower bound on this domain. In other words, $|\sigma_P(\lambda)| \gg 1$ when $|\lambda| = 1$, which implies $|\sigma_P(\lambda)| \gg |\lambda|^\nu$ for all λ. By the triangle inequality, this implies

$$|P(\lambda)| \gg \Big(1 - \frac{|P(\lambda) - \sigma_P(\lambda)|}{|\lambda|^\nu}\Big)|\lambda|^\nu. \tag{22}$$

Since $P(\lambda) - \sigma_P(\lambda)$ is a polynomial of order less than ν, the ratio term is $\ll 1$ when $|\lambda|$ is sufficiently large. So for large $|\lambda|$ we have $|P(\lambda)| \gg |\lambda|^\nu \gg (1+|\lambda|^2)^{\nu/2}$ as required.

The above proof relies on the continuity of the function $\sigma_P(\lambda)$, which is not true in general since λ^α has a branch cut in the complex λ-plane when α is not an integer. But $\sigma_P(\lambda)$ can be approximated arbitrarily closely by a sum of *rational* powers of λ, i.e. a polynomial of order around $m\nu$ in $\lambda^{1/m}$ for some large natural number m. Call this function $\tilde{\sigma}_P(\lambda)$; the above proof shows that $|\tilde{\sigma}_P(\lambda)| \gg 1$ when $|\lambda^{1/m}| = 1$, i.e. when $|\lambda| = 1$. Now by letting the exponents in $\tilde{\sigma}_P$ tend to those in σ_P, we find $|\sigma_P(\lambda)| \gg 1$ when $|\lambda| = 1$, as before. Again this gives equation (1).

Because of the finite bound ϵ mentioned in Definition 2.1.6, the ratio term $\frac{|P(\lambda)-\sigma_P(\lambda)|}{|\lambda|^\nu}$ is still $\ll 1$ for sufficiently large λ, and the result follows. □

Lemma 2.1.9 (Existence of parametrices)**.** *If P(D) is an elliptic fractional partial differential operator as above, then it has a parametrix, i.e. $E \in \mathcal{D}'(\mathbb{R}^n)$ such that $P(D)E = \delta_0 + \omega$ for some $\omega \in \mathcal{E}(\mathbb{R}^n)$, and the parametrix E is in $\mathcal{S}'(\mathbb{R}^n)$ and also in $C^\infty(\mathbb{R}^n \backslash \{0\})$.*

Proof. Fix a test function $\chi \in \mathcal{D}(\mathbb{R}^n)$ which is identically 1 on the domain $|\lambda| \leq R$ and identically 0 on the domain $|\lambda| > R+1$, where R is as in Lemma 2.1.8. Let

$$\hat{E}(\lambda) \coloneqq \frac{1-\chi(\lambda)}{P(\lambda)}.$$

This is well-defined because $1-\chi$ is zero at all zeros of P, and it is bounded by Lemma 2.1.8. By definition of P, we therefore have the leftmost of the following inclusions, leading to the rightmost:

$$\hat{E} \in \mathcal{E}'(\mathbb{R}^n) \Rightarrow \hat{E} \in \mathcal{S}'(\mathbb{R}^n) \Rightarrow E \in \mathcal{S}'(\mathbb{R}^n) \Rightarrow E \in \mathcal{D}'(\mathbb{R}^n),$$

where E is the inverse Fourier transform of $\hat{E}$. Similarly,

$$\chi \in \mathcal{D}(\mathbb{R}^n) \Rightarrow \chi \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \omega \in \mathcal{S}(\mathbb{R}^n) \Rightarrow \omega \in \mathcal{E}(\mathbb{R}^n),$$

where ω is the inverse Fourier transform of $-\chi$. Finally,

$$P(\lambda)\hat{E}(\lambda) = 1 - \chi(\lambda) \Rightarrow P(D)E = \delta_0 + \omega,$$

so E is a parametrix of $P(D)$.

On the domain $|\lambda| > R+1$, we have

$$\left|\widehat{D^\alpha(x^\beta E)}(\lambda)\right| = \left|\lambda^\alpha D^\beta E(\lambda)\right| = \left|\lambda^\alpha D^\beta\big(P(\lambda)^{-1}\big)\right| \ll \left|\lambda\right|^{|\alpha|-|\beta|-\nu}$$

for any multi-indices α, β. So for all α, β with $|\beta|$ sufficiently large, the function $\widehat{D^\alpha(x^\beta E)}$ is in $L^1(\mathbb{R}^n)$, which means its inverse Fourier transform $D^\alpha(x^\beta E)$ is in $C(\mathbb{R}^n)$. So E is in $C^\infty(\mathbb{R}^n\backslash\{0\})$. And the fact that $E \in \mathcal{S}'(\mathbb{R}^n)$ was already established above. $\square$

Lemma 2.1.10. *If $\phi \in \mathcal{D}(\mathbb{R}^n)$ and $u \in H^t(\mathbb{R}^n) \cap \Phi'(\mathbb{R}^n)$ for some $n \in \mathbb{N}, t \in \mathbb{R}$, then $[D^\alpha, \phi](u) \in H^{t-|\alpha|+1}(\mathbb{R}^n)$ for any $\alpha \in \mathbb{C}^n$, where $[,]$ denotes a commutator.*

Proof. Note that when α is an ordinary multi-index in $(\mathbb{Z}_0^+)^n$, this result is straightforwardly proved using the product rule: the operator $[D^\alpha, \phi]$ is just an $(|\alpha|-1)$th-order differential operator. In the general case, however, we need to use infinite series and some more complicated estimates. It may appear that the fractional product rule (Lemma 1.1.8) is applicable, but of course analyticity is out of the question when we are dealing with test functions $\phi \in \mathcal{D}(\mathbb{R}^n)$.

The property of a function f being in a Sobolev space $H^s(\mathbb{R}^n)$ depends only on the large-λ behaviour of the Fourier transform $\hat{f}(\lambda)$, so it will suffice to prove that the Fourier transform of $[D^\alpha, \phi](u)$ behaves like the Fourier transform of a function in $H^{t-|\alpha|+1}(\mathbb{R}^n)$ when $|\lambda|$ has some fixed lower bound.

Firstly, we rewrite the expression as follows:

$$\begin{aligned}\widehat{[D^\alpha, \phi](u)}(\lambda) &= \widehat{D^\alpha(\phi u)}(\lambda) - (\widehat{\phi D^\alpha u})(\lambda) = \lambda^\alpha \hat{\phi}(\lambda) * \hat{u}(\lambda) - \hat{\phi}(\lambda) * (\lambda^\alpha \hat{u}(\lambda)) \\ &= \lambda^\alpha \int_{\mathbb{R}^n} \hat{\phi}(\mu)\hat{u}(\lambda-\mu)\,\mathrm{d}\mu \; - \int_{\mathbb{R}^n} \hat{\phi}(\mu)(\lambda-\mu)^\alpha \hat{u}(\lambda-\mu)\,\mathrm{d}\mu \\ &= I_1(\lambda) + I_2(\lambda),\end{aligned}$$

where the two integral expressions I_1, I_2 are defined by

$$I_1(\lambda) \coloneqq \lambda^\alpha \int_{|\mu|\le\frac{1}{2}|\lambda|} \hat{\phi}(\mu)\Big(1-\big(1-\tfrac{\mu}{\lambda}\big)^\alpha\Big)\hat{u}(\lambda-\mu)\,\mathrm{d}\mu;$$
$$I_2(\lambda) \coloneqq \int_{|\mu|>\frac{1}{2}|\lambda|} \hat{\phi}(\mu)\Big(\lambda^\alpha-(\lambda-\mu)^\alpha\Big)\hat{u}(\lambda-\mu)\,\mathrm{d}\mu.$$

We shall evaluate I_1 and I_2 separately and prove bounds to establish that each of them is the Fourier transform of a function in $H^{t-|\alpha|+1}(\mathbb{R}^n)$, which will suffice to prove the lemma.

Firstly,

$$\begin{aligned}
I_1(\lambda) &= \lambda^\alpha \int_{|\mu|\le\frac{1}{2}|\lambda|} \hat{\phi}(\mu)\left[\sum_{m=1}^{\infty}\tbinom{\alpha}{m}\big(\tfrac{\mu}{\lambda}\big)^m\right]\hat{u}(\lambda-\mu)\,\mathrm{d}\mu \\
&= \lambda^\alpha \int_{|\mu|\le\frac{1}{2}|\lambda|} \hat{\phi}(\mu)\Big[\tbinom{\alpha}{1}\tfrac{\mu}{\lambda}+o\big(\tfrac{\mu}{\lambda}\big)\Big]\hat{u}(\lambda-\mu)\,\mathrm{d}\mu \\
&\sim \alpha\lambda^{\alpha-1}\int_{|\mu|\le\frac{1}{2}|\lambda|} \mu\hat{\phi}(\mu)\hat{u}(\lambda-\mu)\,\mathrm{d}\mu \\
&\ll \alpha\lambda^{\alpha-1}\widehat{\phi'}(\lambda)*\hat{u}(\lambda) \\
&= \alpha\widehat{D^{\alpha-1}(\phi' u)}.
\end{aligned}$$

Since $\phi' \in \mathcal{D}(\mathbb{R}^n)$, we have $\phi' u \in H^t(\mathbb{R}^n)$. By Lemma 1.1.4, this means the above expression is the Fourier transform of a function in $H^{t-|\alpha|+1}(\mathbb{R}^n)$, as required.

Now consider I_2. By the Paley-Wiener-Schwartz theorem (see [77, Chapter 1]), the function $\hat{\phi}$ is entire and satisfies an inequality of the form $|\hat{\phi}(\lambda)| \ll_N (1+|\lambda|)^{-N}$ for $N\in\mathbb{N}$, $\lambda\in\mathbb{R}^n$, where the subscript means the multiplicative constant depends on N. So

$$\begin{aligned}
I_2 &= \int_{|\mu|>\frac{1}{2}|\lambda|} \hat{\phi}(\mu)\Big(\lambda^\alpha-(\lambda-\mu)^\alpha\Big)\hat{u}(\lambda-\mu)\,\mathrm{d}\mu \\
&\ll_N \int_{|\mu|>\frac{1}{2}|\lambda|} (1+|\mu|)^{-N-|\alpha|}\big(|2\mu|^{|\alpha|}+|3\mu|^{|\alpha|}\big)|\hat{u}(\lambda-\mu)|\,\mathrm{d}\mu \\
&\ll \int_{|\mu|>\frac{1}{2}|\lambda|} (1+|\mu|)^{-N}|\hat{u}(\lambda-\mu)|\,\mathrm{d}\mu \\
&\ll \big((1+|\bullet|)^{-N}*|\hat{u}|\big)(\lambda).
\end{aligned}$$

Since u is in $H^t(\mathbb{R}^n)$ and N can be arbitrarily large, this final expression must be the Fourier transform of a function in $H^{t+K}(\mathbb{R}^n)$ for arbitrarily large K. And $H^a\subset H^b$ for $a>b$, so I_2 is the Fourier transform of a function in $H^{t-|\alpha|+1}(\mathbb{R}^n)$, as required. $\square$

Lemma 2.1.11. *If f and g are functions, at least one of which is a Schwartz function, and $\nu\in\mathbb{C}$ is such that ${}_{-\infty}D^\nu f$ and ${}_{-\infty}D^\nu g$ are well-defined, then ${}_{-\infty}D^\nu f*g = f*{}_{-\infty}D^\nu g$,* *where $*$ denotes convolution.*

Proof. When $\mathrm{Re}(\nu) < 0$, writing ${}_{-\infty}D^\nu f = f * \Phi$ as in Lemma 1.1.4 and using the associativity of convolution gives

$$ {}_{-\infty}D^\nu f * g = (f * \Phi) * g = f * (\Phi * g) = f * (g * \Phi) = f * {}_{-\infty}D^\nu g. $$

When $\mathrm{Re}(\nu) \geq 0$, we write $n \coloneqq \lfloor \mathrm{Re}(\nu) \rfloor + 1$ and assume without loss of generality that g is a Schwartz function. Using the definition of Riemann–Liouville derivatives together with the above result gives

$$ {}_{-\infty}D_x^\nu f * g = \left(\tfrac{d^n}{dx^n} \left({}_{-\infty}D_x^{\nu-n} f \right) \right) * g = {}_{-\infty}D_x^{\nu-n} f * \tfrac{d^n g}{dx^n} = f * {}_{-\infty}D_x^{\nu-n} \left(\tfrac{d^n g}{dx^n} \right). $$

The final expression on the right-hand side is a Caputo derivative and not a Riemann–Liouville derivative of g. However, since g is a Schwartz function, its Caputo and Riemann–Liouville derivatives are identical (by the discussion following Lemma 1.1.7), and the result follows. □

Theorem 2.1.12 (Fractional elliptic regularity theorem). *If $P(D)$ is a νth-order elliptic fractional partial differential operator as above and X is a domain in $\mathbb{R}^n$ and $u, f \in \mathcal{D}'(X)$ satisfy $u \in \Phi'(X)$ and $P(D)u = f$, then*

$$ f \in H^s_{loc}(X) \Rightarrow u \in H^{s+\nu}_{loc}(X). $$

Proof. First assume $X = \mathbb{R}^n$ and u is compactly supported (i.e. in $\mathcal{E}'(\mathbb{R}^n)$). By Lemma 2.1.9, $P(D)$ has a parametrix E and (using Lemma 2.1.11)

$$ u = \delta_0 * u = (P(D)E) * u - \omega * u = E * (P(D)u) - \omega * u = E * f - \omega * u. $$

Since u has compact support, $\omega * u$ is a Schwartz function, so it will be enough to prove $E * f \in H^{s+\nu}(\mathbb{R}^n)$. If $|\lambda| > R + 1$, then by Lemma 2.1.8 and the definition of $\hat{E}$,

$$ \left| \widehat{E * f}(\lambda) \right| = \left| \tfrac{\hat{f}(\lambda)}{P(\lambda)} \right| \ll (1 + |\lambda|^2)^{-\nu/2} \hat{f}(\lambda). $$

And $f \in H^s(\mathbb{R}^n)$, so $E * f \in H^{s+\nu}(\mathbb{R}^n)$ as required.

To prove the general case, we shall use a bootstrapping argument. First of all, let us note that it makes sense to define fractional derivatives of functions in $\mathcal{D}'(X)$ even when X does not extend to negative infinity: the integrals from $-\infty$ to x required by Definition 1.1.1 are simply taken to be zero outside of X. In other words, the arbitrary test function $\phi \in \mathcal{D}(X)$ is extended to a function on all of $\mathbb{R}^n$ which is supported on X.

Fix $\phi \in \mathcal{D}(X)$; it will suffice to prove that $\phi u \in H^{s+\nu}(\mathbb{R}^n)$. Let $\psi_0, \psi_1, \ldots, \psi_m$ (where the value of m will be decided later) be test functions in $\mathcal{D}(\mathbb{R}^n)$ with supports as shown

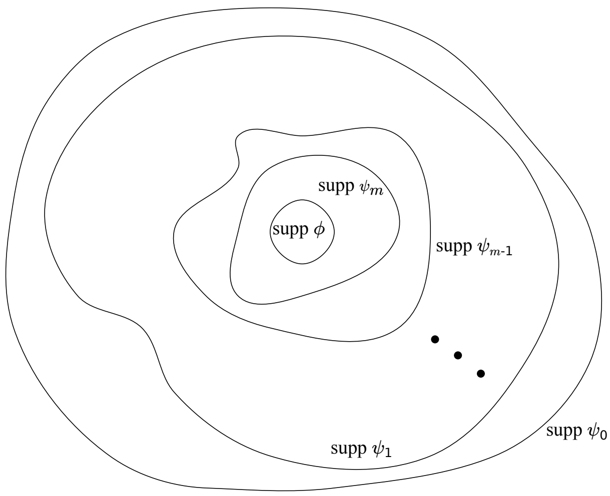

Figure 1: The domains involved in the bootstrapping proof of Theorem 2.1.12

in Figure 1, i.e. such that:

$$\begin{aligned} \operatorname{supp}(\phi) \subset \operatorname{supp}(\psi_m), \quad \psi_m = 1 \text{ on } \operatorname{supp}(\phi); \\ \operatorname{supp}(\psi_i) \subset \operatorname{supp}(\psi_{i-1}), \quad \psi_{i-1} = 1 \text{ on } \operatorname{supp}(\psi_i) \quad \forall i. \end{aligned} \tag{23}$$

Now $\psi_0 u$ is in $\mathcal{E}'(\mathbb{R}^n)$ and therefore in $H^t(\mathbb{R}^n)$ for some $t \in \mathbb{R}$. So

$$\begin{aligned} P(D)(\psi_1 u) &= \psi_1 P(D)u + [P(D), \psi_1]u && \text{(where } [,] \text{ is a commutator)} \\ &= \psi_1 f + [P(D), \psi_1](\psi_0 u) && \text{(by (23))} \\ &= \Big(\in H^s(\mathbb{R}^n) \Big) + \Big(\in H^{t-\nu+1)}(\mathbb{R}^n) \Big) && \text{(by Lemma 2.1.10)} \\ &\in H^{\min(s,t-\nu+1)}(\mathbb{R}^n) && \text{(since } a > b \Rightarrow H^a \subset H^b\text{).} \end{aligned}$$

From here we can use the first part of the proof to deduce that $\psi_1 u \in H^{A_1}(\mathbb{R}^n)$ where $A_1 := \min(s, t - \nu + 1) + \nu = \min(s + \nu, t + 1)$.

By exactly the same argument, $P(D)(\psi_2 u) = \psi_2 f + [P(D), \psi_2](\psi_1 u)$ and $\psi_2 u \in H^{A_2}(\mathbb{R}^n)$ where $A_2 := \min(s + \nu, A_1 + 1) = \min(s + \nu, t + 2)$.

Continuing in this manner eventually yields $\psi_m u \in H^{\min(s+\nu,t+m)}(\mathbb{R}^n)$. Now we set the natural number m to be $\lceil s + \nu - t \rceil + 1$, so that $\psi_m u \in H^{s+\nu}(\mathbb{R}^n)$, which means $\phi u \in H^{s+\nu}(\mathbb{R}^n)$ as required, by (23). $\square$

2.1.3 Examples and extensions

As example applications of our work, we consider the following two simple corollaries.

Corollary 2.1.13. *Let $P(D)$ be a fractional linear partial differential operator of the form described above. If it is elliptic, then it is also hypoelliptic.*

Proof. Recall the definition of hypoellipticity: a partial differential operator ∂ is hypoelliptic if whenever ∂u is a smooth function, so also is u on the same domain.

If $P(D)$ is elliptic, then using all notation as in Theorem 2.1.12, we must have $f \in C^\infty(X) \Rightarrow u \in C^\infty(X)$, i.e. $P(D)$ is also hypoelliptic. $\square$

Corollary 2.1.14. *Consider the operator $\widetilde{\Delta}_\alpha \coloneqq \sum_{i=1}^n \partial_i^\alpha$ with $0 < \alpha < 1$, a fractional generalisation of the Laplacian, and a function $u \in \mathcal{D}'(X)$ where X is a domain in $\mathbb{R}^n$.*

If u is a solution to the fractional Laplace-type equation $\widetilde{\Delta}_\alpha u = 0$, then it must necessarily be smooth. More generally, if u is the solution to a fractional Poisson-type equation $\widetilde{\Delta}_\alpha v = f$ with forcing $f \in H^s_{loc}(X)$, then $u \in H^{s+\alpha}_{loc}(X)$.

Proof. The fractional operator $\sum_{i=1}^n \partial_i^\alpha$ is elliptic when $0 < \alpha < 1$, since then λ^α is in the right half complex plane for all $\lambda \in \mathbb{R}$. So Theorem 2.1.12 applies and the results follow. $\square$

The result proved herein is only one of many possible versions of a fractional elliptic regularity theorem.

For classical PDEs, there are far more elliptic regularity theorems than Theorem 2.1.5, which covers only linear partial differential operators whose coefficients are constants in $\mathbb{C}$. Other versions concern linear partial differential operators with non-constant coefficients, perhaps satisfying some C^k or L^p condition; the Sobolev conditions can also sometimes be replaced by L^p conditions on the functions f and u. See e.g. [58, Chapter 6C] and [44, Chapter 6.3]. These other variants of the elliptic regularity theorem may well be extendable to fractional PDEs just as Theorem 2.1.5 was.

Furthermore, there are more models of fractional calculus than just the Riemann–Liouville formula (several of which are discussed in depth below). Some of them cooperate with the Fourier transform almost as well as Riemann–Liouville differintegrals do, which was a necessary factor in our proofs here. Thus, with a little more work we may be able to prove results analogous to Theorem 2.1.12 for fractional PDEs defined using other fractional models, which have different applications from the Riemann–Liouville one. Work in this area is currently ongoing.

2.2 The Malgrange–Ehrenpreis theorem

2.2.1 Introduction

Another important result in PDE theory is the **Malgrange–Ehrenpreis theorem**, which guarantees the existence of a fundamental solution for any linear partial differential operator with constant coefficients. This is significant because the solution to a given PDE with arbitrary forcing can be generated from the solution with delta-function forcing, i.e. from the fundamental solution, by using convolution of functions. So by proving the existence of fundamental solutions, the Malgrange–Ehrenpreis theorem guarantees the existence of solutions to any linear PDE with constant coefficients on the derivative terms and arbitrary forcing.

Here we seek to extend this theorem in order to find fundamental solutions for fractional partial differential operators defined using the Riemann–Liouville formula.

Theorem 2.2.1 (Malgrange–Ehrenpreis theorem). *Every non-zero linear constant-coefficient partial differential operator, i.e. every operator $P(D)$ where P is a complex n-variable Nth-order polynomial and $D \coloneqq -i\frac{\partial}{\partial x}$ for the n-dimensional variable $x \in \mathbb{R}^n$, has a fundamental solution, i.e. a distribution $E \in \mathcal{D}'(\mathbb{R}^n)$ such that $P(D)E = \delta_0$.*

The original proofs of this result by Malgrange [100] and Ehrenpreis [41], who proved it independently, were non-constructive and used the Hahn–Banach theorem. But several constructive proofs have since been devised, and some of these can be extended to certain subcases of the fractional context in order to prove generalisations of the theorem.

2.2.2 Proof using Hörmander staircases

One semi-constructive proof due to Hörmander [76] involves building a solution by using complex integration over a Hörmander staircase.

This proof, like many others, relies on the fact that P is a polynomial: we need to use the Fundamental Theorem of Algebra to factorise $P(\lambda)$ into linear terms of the form $\lambda_n - f_j(\lambda_1, \ldots, \lambda_{n-1})$ and then analyse these linear factors. If $P(\lambda)$ is a general function of the form $\sum_\alpha c_\alpha \lambda^\alpha$ where the exponents α may be real or complex, then the Fundamental Theorem of Algebra no longer applies.

However, if all the α are *rational*, the proof can be modified so that it still works. Instead of considering the differential operator $P(D)$ as a polynomial in D, we can consider it as a polynomial in $D^{1/K}$ for some sufficiently large natural number K, factorise this polynomial using the Fundamental Theorem of Algebra, and proceed more or less as before. In fact, since Hörmander's proof never uses the fact that $P(D)$ is a polynomial in $D_1, D_2, \ldots, D_{n-1}$, it will suffice to assume only that (for example) all the final components α_n are rational, as in the following theorem.

Theorem 2.2.2 (Malgrange–Ehrenpreis theorem: rational-order derivatives). *Let $P(\lambda)$ be a function of the complex n-dimensional parameter λ of the form $\sum_\alpha c_\alpha \lambda^\alpha$ where the sum is finite, the multi-indices α are in $(\mathbb{R}^+)^n$, and there exists j such that all the jth coordinates α_j of the multi-indices are in $\mathbb{Q}$. If $x \in \mathbb{R}^n$ is an n-dimensional variable and powers of $D := -i\frac{\partial}{\partial x}$ are defined using the Riemann–Liouville formula with $c = -\infty$, then the partial differential operator $P(D)$ has a fundamental solution, i.e. a distribution $E \in \mathcal{D}'(\mathbb{R}^n)$ such that $P(D)E = \delta_0$.*

Proof. Without loss of generality, say $j = n$. Let λ' denote the vector $(\lambda_1, \lambda_2, \dots, \lambda_{n-1})$ in $\mathbb{R}^{n-1}$, and note that $P(\lambda) = P(\lambda', \lambda_n)$ can be written as a polynomial in $\lambda_n^{1/K}$ with coefficients depending on λ', where K is a fixed natural number (the LCM of the denominators of the exponents α_n). More explicitly, write

$$P(\lambda', \lambda_n) = A(\lambda')\Big(\lambda_n^{M/K} + \sum_{j=0}^{M-1} a_j(\lambda')\lambda_n^{j/K}\Big)$$

where M is a natural number and the A, a_j are continuous functions of $\lambda' \in \mathbb{R}^{n-1}$. In particular, $A(\lambda')$ is a product of power functions $\lambda_i^{\alpha_i}$, so $A(\lambda') = 0$ only if $\lambda' = 0$. By the Fundamental Theorem of Algebra, $P(\lambda)$ can then be written as

$$P(\lambda', \lambda_n) = A(\lambda')\prod_{j=1}^{M}\Big(\lambda_n^{1/K} - \tau_j(\lambda')\Big)$$

where the τ_j are continuous functions on $\mathbb{R}^{n-1}$. (If λ' were allowed to be complex, there would be complications with branch cuts, but as it is real, the A, a_j, τ_j can be defined to be continuous.)

Fix $\mu \in \mathbb{R}^{N-1}\backslash\{0\}$; we wish to bound $P(\mu, \lambda_n)$ below, in order to get an upper bound on its reciprocal. Now let $R = R(\mu) := \max_j |\tau_j(\mu)| + |A(\mu)|^{-1/M} + 1$ (this is in $\mathbb{R}^+$ since $\mu \neq 0 \Rightarrow A(\mu) \neq 0$). By continuity of the A, τ_j, there exists an open neighbourhood $N(\mu) \subset \mathbb{R}^{n-1}\backslash\{0\}$ of μ such that for all $\lambda' \in N(\mu)$, $\max_j |\tau_j(\lambda')| + |A(\lambda')|^{-1/M} < R$. Now whenever $\left|\lambda_n^{1/K}\right| \geq R(\mu)$ and $\lambda' \in N(\mu)$, we have

$$\left|\lambda_n^{1/K} - \tau_j(\lambda')\right| > \left|A(\lambda')\right|^{-1/M}$$

for each j, and therefore

$$\left|P(\lambda', \lambda_n)\right| = \left|A(\lambda')\right|\prod_{j=1}^{M}\left|\lambda_n^{1/K} - \tau_j(\lambda')\right| > \left|A(\lambda')\right|\prod_{j=1}^{M}\left|A(\lambda')\right|^{-1/M} = 1.$$

In particular, define $\gamma = \gamma(\mu)$ for $\mu \in \mathbb{R}^{n-1}$ to be the black contour shown in Figure 2, i.e.

$$\gamma = \{re^{i\pi} : \infty > r > R^K\} \cup \{R^K e^{i\theta} : -\pi < \theta < \pi\} \cup \{r : R^K < r < \infty\}.$$

Since $\lambda_n^{1/K}$ is on the red contour shown in Figure 2 when λ_n is on the black one, we have $\left|\lambda_n^{1/K}\right| \geq R$ for all $\lambda_n \in \gamma$ and therefore

$$\left|P(\lambda', \lambda_n)\right| > 1 \text{ for } \lambda' \in N(\mu), \lambda \in \gamma(\mu). \tag{24}$$

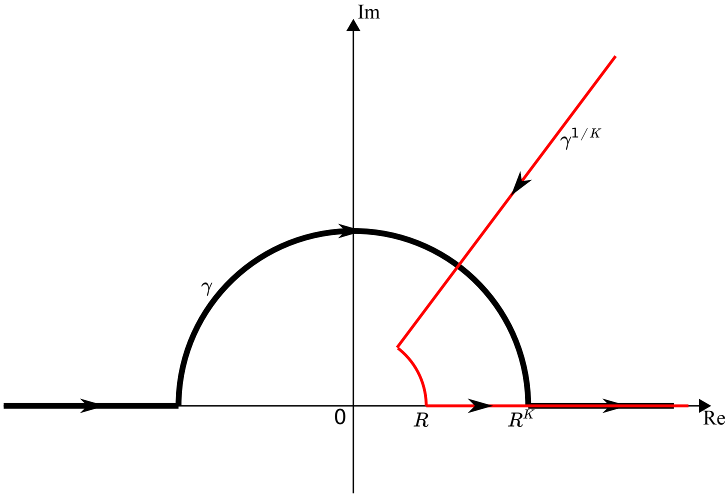

Figure 2: The contours for γ and $\gamma^{1/K}$

The sets $N(\mu)$ form an open cover of $\mathbb{R}^{n-1}\backslash\{0\}$. But $\mathbb{R}^{n-1}\backslash\{0\}$ is an open subset of $\mathbb{R}^{n-1}$ and therefore locally compact, and it is also σ-compact, so it must be a Lindelöf space, i.e. every open cover has a countable subcover. So there is a countable sequence $\mu_1, \mu_2, \mu_3, \dots$ such that the open sets $N(\mu_k)$ cover $\mathbb{R}^{n-1}\backslash\{0\}$. Let $\Delta_k \coloneqq N(\mu_k)\backslash \bigcup_{j=1}^{k-1} \overline{N(\mu_j)}$ for all k; these sets are open and disjoint and $\bigcup_{k=1}^{\infty} \overline{\Delta_k} = \mathbb{R}^{n-1}$.

Define $E \in \mathcal{D}'(\mathbb{R}^n)$ by

$$\langle E, \phi \rangle = (2\pi)^{-n} \sum_{k=1}^{\infty} \int_{\Delta_k} \int_{\gamma(\mu_k)} \frac{\hat{\phi}(-\lambda', -\lambda_n)}{P(\lambda', \lambda_n)} \, \mathrm{d}\lambda_n \, \mathrm{d}\lambda' \,; \tag{25}$$

this is well-defined as a distribution, since (24) tells us that $|P| > 1$ on all regions integrated over. Now for any $\phi \in \mathcal{D}(\mathbb{R}^n)$, (25) implies

$$\begin{aligned}\langle P(D)E, \phi\rangle = \langle E, P(D)\phi\rangle &= (2\pi)^{-n} \sum_{k=1}^{\infty} \int_{\Delta_k} \int_{\gamma(\mu_k)} \frac{\widehat{P(D)\phi}(-\lambda', -\lambda_n)}{P(\lambda', \lambda_n)} \, \mathrm{d}\lambda_n \, \mathrm{d}\lambda' \\ &= (2\pi)^{-n} \sum_{k=1}^{\infty} \int_{\Delta_k} \int_{\gamma(\mu_k)} \hat{\phi}(-\lambda', -\lambda_n) \, \mathrm{d}\lambda_n \, \mathrm{d}\lambda' \\ &= (2\pi)^{-n} \sum_{k=1}^{\infty} \int_{\Delta_k} \int_{\mathbb{R}} \hat{\phi}(-\lambda', -\lambda_n) \, \mathrm{d}\lambda_n \, \mathrm{d}\lambda' \\ &= (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \hat{\phi}(-\lambda', -\lambda_n) \, \mathrm{d}\lambda_n \, \mathrm{d}\lambda' = \phi(0),\end{aligned}$$

so $P(D)E = \delta_0$ as required. (To get from the second line to the third above, we used Cauchy's theorem and the fact that the Fourier transform $\hat{\phi}(\lambda)$ of a test function $\phi \in \mathcal{D}(\mathbb{R}^n)$ is analytic in each coordinate of λ.) □

Remark 2.2.3. In the proof of Theorem 2.2.2, the argument is roughly based on that of Hörmander [76], with the important difference that we need to consider the function $\lambda_n^{1/K}$ as well as just λ_n. This makes things more complicated at a few points in the proof.

For one thing, since we can no longer make a linear change of coordinates in the vector variable λ, we now need to account for the function $A(\lambda)$ in our estimates, whereas in the non-fractional proof this function could be assumed without loss of generality to be constant. For another, we need to be more careful about the bounds we set on the variables variables λ_n and $\lambda_n^{1/K}$, and we end up looking at both of the two curves shown in Figure 2 rather than just a single type of curve as in the non-fractional proof.

Since any real-order differintegral operator can be approximated arbitrarily closely by rational-order ones, Theorem 2.2.2 is sufficient to get accurate numerical approximations to fundamental solutions of any non-zero linear constant-coefficient fractional partial differential operator which contains only *real*-order differintegrals. So from the point of view of real-world applications, we have got as far as necessary with this theorem.

2.2.3 Proof using Wagner construction

A more recent proof of the Malgrange–Ehrenpreis theorem due to Ortner and Wagner [110] involves constructing explicit fundamental solutions using inverse Fourier transforms.

In this proof, the fact that P is a polynomial is relevant because the binomial theorem is used to turn an expression of the form $P(\partial + \lambda\eta)$ into a finite sum, and also because the residue theorem is used to cancel out most terms in this finite sum. The binomial theorem, in a more complicated form involving *infinite* series, can still be applied when P is not a polynomial; the residue theorem is harder to apply in this case, and so we again require an extra assumption.

Theorem 2.2.4 (**Malgrange–Ehrenpreis theorem: real-order derivatives with integer differences**). *Let $P(\lambda)$ be a function of the complex n-dimensional parameter λ of the form $\sum_\alpha c_\alpha \lambda^\alpha$ where the sum is finite, the multi-indices α are in $(\mathbb{R}^+)^n$, and there exists $A \in \mathbb{R}$ such that all the magnitudes $|\alpha| = \Sigma_j \alpha_j$ of the multi-indices are of the form $A - m$ for some integer $m \geq 0$. If $x \in \mathbb{R}^n$ is an n-dimensional variable and powers of $D \coloneqq -i\frac{\partial}{\partial x}$ are defined using the Riemann–Liouville formula with $c = -\infty$, then the partial differential operator $P(D)$ has a fundamental solution, i.e. a distribution $E \in \mathcal{D}'(\mathbb{R}^n)$ such that $P(D)E = \delta_0$.*

Proof. We define the fundamental solution E by

$$E(x) \coloneqq \frac{1}{2\pi i \overline{P_A(-i\eta)}} \int_{S^1} \lambda^{A-1} e^{\lambda \eta x} \psi_\lambda(x) \, \mathrm{d}\lambda$$

where $\eta \in \mathbb{R}^n$ is a fixed real vector, $P_A(\lambda) \coloneqq \sum_{|\alpha|=A} c_\alpha \lambda^\alpha$ is the 'maximum order' part of P, and the Schwartz distribution ψ_λ is defined by its Fourier transform being

$$\widehat{\psi_\lambda}(\xi) = \frac{\overline{P(\xi - i\lambda\eta)}}{P(\xi - i\lambda\eta)}.$$

Now there are two things we need to prove: firstly that E is a well-defined distribution, and secondly that $P(D)E = \delta_0$.

Firstly, note that the zero set of P in $\mathbb{R}^n$ has Lebesgue measure zero, so $\frac{\overline{P(\xi-i\lambda\eta)}}{P(\xi-i\lambda\eta)}$ is an L^∞ function of $\xi \in \mathbb{R}^n$, and therefore a Schwartz distribution, for any fixed $\lambda \in \mathbb{C}, \eta \in \mathbb{R}$. So $\psi_\lambda \in \mathcal{S}'(\mathbb{R}^n)$ is well-defined. Also the map

$$\begin{aligned} S^1 &\to \mathcal{S}'(\mathbb{R}^n) \\ \lambda &\mapsto \tfrac{\overline{P(\xi-i\lambda\eta)}}{P(\xi-i\lambda\eta)} \end{aligned}$$

is continuous. So E is the integral over a compact set of a continuous function taking values in $\mathcal{D}'(\mathbb{R}^n)$, and therefore well-defined as an element of $\mathcal{D}'(\mathbb{R}^n)$.

Let us use $\mathcal{F}$ to denote the Fourier transform from variable $x \in \mathbb{R}^n$ to variable $\xi \in \mathbb{R}^n$, so that $\psi_\lambda(x) = \mathcal{F}^{-1}\left(\frac{\overline{P(\xi-i\lambda\eta)}}{P(\xi-i\lambda\eta)}\right)$. Now consider how the fractional partial differential operator $P(D)$ works on a function of the form $e^{\lambda\eta x}\mathcal{F}^{-1}S$ for some Schwartz distribution

S. By the fractional product rule, Lemma 1.1.8, we have

$$\begin{aligned}
D^\alpha\big(e^{\lambda\eta x}\mathcal{F}^{-1}S\big) &= (-i)^\alpha{}_{-\infty}D_x^\alpha\big(e^{\lambda\eta x}\mathcal{F}^{-1}S\big)\\
&= (-i)^\alpha \sum_k \binom{\alpha}{k}{}_{-\infty}D_x^{\alpha-k}\big(e^{\lambda\eta x}\big){}_{-\infty}D_x^k\big(\mathcal{F}^{-1}S\big)\\
&= e^{\lambda\eta x}\sum_k \binom{\alpha}{k}\big(-i\lambda\eta\big)^{\alpha-k}D^k\big(\mathcal{F}^{-1}S\big)\\
&= e^{\lambda\eta x}\sum_k \binom{\alpha}{k}\big(-i\lambda\eta\big)^{\alpha-k}\mathcal{F}^{-1}\big(\xi^k S\big)\\
&= e^{\lambda\eta x}\mathcal{F}^{-1}\Big(\sum_k \tbinom{\alpha}{k}\big(-i\lambda\eta\big)^{\alpha-k}\xi^k S\Big)\\
&= e^{\lambda\eta x}\mathcal{F}^{-1}\left(\xi^\alpha\sum_k \tbinom{\alpha}{k}\Big(\tfrac{-i\lambda\eta}{\xi}\Big)^{\alpha-k}S\right)\\
&= e^{\lambda\eta x}\mathcal{F}^{-1}\left(\xi^\alpha\Big(1+\tfrac{-i\lambda\eta}{\xi}\Big)^\alpha S\right) = e^{\lambda\eta x}\mathcal{F}^{-1}\Big(\big(\xi-i\lambda\eta\big)^\alpha S\Big)
\end{aligned}$$

for any multi-index $\alpha \in (\mathbb{R}_0^+)^n$, where the sums are taken over all multi-indices $k \in (\mathbb{Z}_0^+)^n$, and where we use Lemma 1.1.4 between the third and fourth lines. So by finite summation, it follows that

$$P(D)\big(e^{\lambda\eta x}\mathcal{F}^{-1}S\big) = e^{\lambda\eta x}\mathcal{F}^{-1}\big(P(\xi-i\lambda\eta)S\big).$$

In particular, setting $S(\xi) = \widehat{\psi_\lambda}(\xi) = \frac{\overline{P(\xi-i\lambda\eta)}}{P(\xi-i\lambda\eta)}$ gives

$$\begin{aligned}
P(D)\big(e^{\lambda\eta x}\psi_\lambda(x)\big) &= e^{\lambda\eta x}\mathcal{F}^{-1}\big(\overline{P(\xi-i\lambda\eta)}\big) = e^{\lambda\eta x}\mathcal{F}^{-1}\big(\overline{P}(\xi+i\overline{\lambda}\eta)\big)\\
&= e^{\lambda\eta x}\mathcal{F}^{-1}\Big(\sum_\alpha \overline{c_\alpha}(\xi+i\overline{\lambda}\eta)^\alpha\Big)\\
&= e^{\lambda\eta x}\mathcal{F}^{-1}\Big(\sum_\alpha \overline{c_\alpha}\sum_k \tbinom{\alpha}{k}\big(i\overline{\lambda}\eta\big)^{\alpha-k}\xi^k\Big)\\
&= e^{\lambda\eta x}\sum_\alpha \overline{c_\alpha}\sum_k \tbinom{\alpha}{k}\big(i\overline{\lambda}\eta\big)^{\alpha-k}D^k\big(\mathcal{F}^{-1}(\mathbb{1})\big)\\
&= e^{\lambda\eta x}\sum_\alpha \overline{c_\alpha}\sum_k \tbinom{\alpha}{k}\big(i\overline{\lambda}\eta\big)^{\alpha-k}\delta_0^{(k)}(x).
\end{aligned}$$

Putting this together with the formula for E, and using the fact that λ is a scalar of modulus 1, we get

$$
\begin{aligned}
P(D)E &= \frac{1}{2\pi i\overline{P_A(-i\eta)}}\int_{S^1}\lambda^{A-1}P(D)\big(e^{\lambda\eta x}\psi_\lambda(x)\big)\mathrm{d}\lambda \\
&= \frac{1}{2\pi i\overline{P_A(-i\eta)}}\int_{S^1}\lambda^{A-1}e^{\lambda\eta x}\sum_\alpha\overline{c_\alpha}\sum_k\big(\overline{i\lambda}\big)^{|\alpha|-|k|}\tbinom{\alpha}{k}\eta^{\alpha-k}\delta_0^{(k)}(x)\,\mathrm{d}\lambda \\
&= \frac{1}{2\pi i\overline{P_A(-i\eta)}}\sum_\alpha\overline{c_\alpha}\sum_k\tbinom{\alpha}{k}(i\eta)^{\alpha-k}\bigg(\int_{S^1}\lambda^{A-1-|\alpha|+|k|}e^{\lambda\eta x}\,\mathrm{d}\lambda\bigg)\delta_0^{(k)}(x).
\end{aligned}
$$

Now we use the hypothesis that all $|\alpha|$ are of the form $A-m$ for non-negative $m\in\mathbb{Z}$. So the residue theorem enables us to eliminate all terms except those where $|\alpha|$ is maximal and $k=0$, resulting in:

$$
\begin{aligned}
P(D)E &= \frac{1}{2\pi i\overline{P_A(-i\eta)}}\sum_{\alpha:|\alpha|=A}\overline{c_\alpha}(i\eta)^\alpha\bigg(\int_{S^1}\lambda^{A-1-A+0}e^{\lambda\eta x}\,\mathrm{d}\lambda\bigg)\delta_0(x) \\
&= \frac{1}{\overline{P_A(-i\eta)}}\sum_{\alpha:|\alpha|=A}\overline{c_\alpha}(i\eta)^\alpha\delta_0(x) = \frac{1}{\overline{P_A(-i\eta)}}\overline{\sum_{\alpha:|\alpha|=A}c_\alpha(-i\eta)^\alpha}\delta_0(x) \\
&= \delta_0(x),.
\end{aligned}
$$

as required. □

Remark 2.2.5. In the proof of Theorem 2.2.4, the argument is roughly based on that of Ortner and Wagner [110], with the important difference that the series throughout the proof which were finite in the non-fractional proof have now become infinite. This is partly because of the fact that when fractional powers are involved, we need to use the more general form of the binomial theorem rather than the simple finite-series form that works for natural-number exponents. It also relates to the fact that we must now use Osler's infinite-series version of the product rule rather than the standard Leibniz rule.

Perhaps the most useful sub-case of Theorem 2.2.4, in which all terms of the partial differential operator must be of order differing by an integer from a fixed number, is where all the terms have the same order, i.e. $P=P_A$.

2.2.4 Examples and extensions

Example 2.2.6. Let us consider $P(D)=\frac{\partial^\alpha}{\partial x^\alpha}$, i.e. the operator given by the power function $P(\lambda)=(i\lambda)^\alpha$, where α is fixed and rational. We shall assume $\alpha>1$, i.e. all components of α are greater than 1, for reasons which will become clear later. Now P is analytic on $\mathbb{C}\backslash\mathbb{R}_0^-$, so the contour $\gamma(\mu)$ can be deformed, regardless of μ, to the same contour with $R=1$. Call this contour γ_1, i.e.

$$\gamma_1=\{re^{i\pi}:\infty>r>1\}\cup\{e^{i\theta}:-\pi<\theta<\pi\}\cup\{r:1<r<\infty\}.$$

So we can write

$$\begin{aligned}\langle E, \phi\rangle &= (2\pi)^{-n} \sum_{k=1}^{\infty} \int_{\Delta_k} \int_{\gamma_1} (i\lambda)^{-\alpha} \hat{\phi}(-\lambda)\, \mathrm{d}\lambda_n\, \mathrm{d}\lambda' \\ &= (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} \int_{\gamma_1} (i\lambda)^{-\alpha} \hat{\phi}(-\lambda)\, \mathrm{d}\lambda_n\, \mathrm{d}\lambda'\end{aligned}$$

as a possibility for the fundamental solution E. By symmetry (or indeed by induction on n), we can therefore define E as follows:

$$\begin{aligned}\langle E, \phi\rangle &= (2\pi)^{-n} \int_{\gamma_1} \cdots \int_{\gamma_1} (i\lambda)^{-\alpha} \hat{\phi}(-\lambda)\, \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_n \\ &= (2\pi)^{-n} \int_{\gamma_1} \cdots \int_{\gamma_1} (i\lambda)^{-\alpha} \Big(\int_{\mathbb{R}^n} e^{i\lambda\cdot x} \phi(x)\, \mathrm{d}x \Big)\, \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_n \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \phi(x) \Big(\int_{\gamma_1} \cdots \int_{\gamma_1} (i\lambda)^{-\alpha} e^{i\lambda\cdot x}\, \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_n \Big)\, \mathrm{d}x,\end{aligned}$$

where Fubini's theorem was used to get from the second line to the third, this being valid because ϕ is a test function, $|\lambda| \geq 1$ for all $\lambda \in \gamma_1$, and $\lambda^{1-\alpha}$ decays at infinity (here we use the assumption that $\alpha > 1$).

So the distribution E can be identified with the function

$$\begin{aligned}E(x) &= (2\pi)^{-n} \int_{\gamma_1} \cdots \int_{\gamma_1} (i\lambda)^{-\alpha} e^{i\lambda\cdot x}\, \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_n \\ &= (2\pi)^{-n} \prod_{k=1}^{n} \Big(\int_{\mathbb{R}} (i\lambda)^{-\alpha_k} e^{i\lambda x_k}\, \mathrm{d}\lambda \Big) \\ &= (2\pi)^{-n} \prod_{k=1}^{n} \Big(\tfrac{2\pi}{\Gamma(\alpha_k)} H(x_k) x_k^{\alpha_k - 1} \Big) = \frac{H(x) x^{\alpha-1}}{\Gamma(\alpha)}\end{aligned}$$

where H is the Heaviside step function defined by $H(x) = 1$ if $x > 0$, $H(x) = 0$ if $x < 0$, and the functions H, Γ applied to the vector variable x are defined by taking the product over the individual coordinates of x. □

Once again, it may be possible to extend the results of this chapter to apply in other fractional models as well as Riemann–Liouville. Note that the only property of Riemann–Liouville fractional derivatives used in the proof of Theorem 2.2.2 is that they work well with Fourier transforms, i.e. the result of Lemma 1.1.4, while the proof of Theorem 2.2.4 also uses the fractional product rule, Lemma 1.1.8. However, both of these results have analogues in several other fractional models, as we shall see later on. Thus it seems likely that versions of the Malgrange–Ehrenpreis theorem can be used in some of these harder models. The resulting formulae for the fundamental solutions will of course be much more complicated, but in many contexts it is the existence result that is most important.

2.3 The unified transform method

2.3.1 Introduction to the method

The **unified transform method**, or **Fokas method**, for solving partial differential equations is a novel technique due to Fokas [53]. It involves integral transforms with respect to both spatial and temporal variables, both types of transform being done simultaneously so that they cannot be considered separately. It is more widely applicable than classical transform methods, being usable even in contexts where these fail, including for certain classes of PDEs with spatial domains such as the half-line and the finite interval. Most importantly, it is *constructive*, generating explicit solutions in integral form for PDEs to which it applies. See also [54] and [55] for more detail about this method and its applicability.

One important context where this method can be applied is in solving equations of the form

$$q_t + w\big(-i\tfrac{\partial}{\partial x}\big)q = 0, \qquad x \in (0,\infty), t \in (0,T), \tag{26}$$

where w is a polynomial function such that $\mathrm{Re}(w(k)) \geqslant 0\ \forall k \in \mathbb{R}$, with initial condition $q(x,0) = q_0(x)$ (for some known function q_0) and appropriate boundary conditions to be fixed later. On equations of this form, the unified transform method works roughly as follows.

Step 1: divergence form. Introducing an exponential term enables us to write the given PDE as a family of PDEs in divergence form, parametrised by a new complex variable k. Specifically, it becomes

$$\big(e^{-ikx+w(k)t}q\big)_t = \big(e^{-ikx+w(k)t}Q\big)_x, \tag{27}$$

where the function $Q(x,t,k)$ must satisfy

$$\left(\frac{\partial}{\partial x} - ik\right)Q = \left(w(k) - w\left(-i\frac{\partial}{\partial x}\right)\right)q.$$

Since w is a polynomial, Q can be defined as a finite series:

$$Q(x,t,k) = i\left(\frac{w(k)-w(l)}{k-l}\right)\Bigg|_{l=-i\frac{\partial}{\partial x}}(q) = \sum_{j=0}^{n-1} c_j(k)\frac{\partial^j q}{\partial x^j}, \tag{28}$$

for some complex polynomials $c_0, c_1, \ldots, c_{n-1}$.

Step 2: global relation. Re-expressed in divergence form, the PDE can now be integrated with respect to both x and t. More specifically, we first substitute τ for t and then apply the two operators $\int_0^\infty \mathrm{d}x$ and $\int_0^t \mathrm{d}\tau$ to (27). On the left-hand side, the t-derivative disappears and we get an x-integral transform, which turns out to be the

Fourier transform. On the right-hand side, the x-derivative disappears and we get a t-integral transform, which is a relative of the Fourier transform but considerably more complicated. The resulting identity is called the **global relation** and is central to the working of the unified transform method:

$$e^{w(k)t}\hat{q}(k,t) = \hat{q}_0(k) - \tilde{g}(k,t), \qquad \mathrm{Im}(k) < 0, t \in (0,T). \tag{29}$$

Here $\hat{\ }$ denotes the Fourier transform and $\tilde{g}(k,t)$ is a much more complicated function, involving t-integral transforms with kernel $e^{w(k)t}$ applied to the functions $\frac{\partial^j q}{\partial x^j}(0,t)$ for $j = 0, 1, \ldots, n-1$.

Step 3: integral formula. The global relation (29) is essentially an expression for the x-Fourier transform of $q(x,t)$ in terms of various initial and boundary data. Applying the Fourier inversion theorem yields an integral expression for $q(x,t)$ itself, as desired:

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-w(k)t}\hat{q}_0(k)\,\mathrm{d}k - \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-w(k)t}\tilde{g}(k,t)\,\mathrm{d}k. \tag{30}$$

Now by Cauchy's theorem, the contour used for the second integral can be deformed from the real line to the boundary of the domain

$$D^+ \coloneqq \{k \in \mathbb{C} : \mathrm{Re}(w(k)) < 0, \mathrm{Im}(k) > 0\}.$$

The choice of the domain D^+ is motivated by considerations of exponential growth and decay. In order for Cauchy's theorem and Jordan's lemma to be applicable, both of the exponential terms e^{ikx} and $e^{-w(k)t}$ appearing in the integrand should decay as $k \to \infty$ in the regions of the complex k-plane through which the contour is deformed.

Step 4: cancelling boundary terms. The formula (30) is not the best result that can be obtained. It expresses q in terms of the given initial data q_0 and boundary data consisting of the functions $\frac{\partial^j q}{\partial x^j}(0,t)$ for $0 \leqslant j < n$, which are considerably more boundary conditions than necessary for the problem to be well-posed. The final step of the unified transform method involves substituting the global relation (29), after appropriate transformations, into the equation (30) and cancelling out some of these boundary terms.

The global relation holds for $\mathrm{Im}(k) < 0$, while the contour of integration ∂D^+ is contained in the upper half plane, so some substitutions will have to be made. We replace k in (29) by $\nu(k)$, where ν is a w-preserving function ($w(\nu(k)) \equiv w(k)$) mapping ∂D^+ into the lower half plane, and then substitute the resulting identity into (30).

In general there will be several possible functions ν to choose from, and the identities resulting from them will give several simultaneous equations in a similar form to (30). From these equations, half of the boundary conditions can then be eliminated, leaving $\sim \frac{n}{2}$ boundary conditions that still need to be fixed at the start.

Example 2.3.1. As a demonstration of the method, let us examine how it would be applied to the following third-order PDE on the half-line:

$$\begin{aligned} q_t + q_{xxx} &= 0, \qquad && x \in (0,\infty), t \in (0,T); \\ q(x,0) &= q_0(x), \qquad && x \in (0,\infty); \\ q(0,t) &= g_0(t), \qquad && t \in (0,T). \end{aligned}$$

We introduce the 'dummy' boundary conditions $g_1(t) = q_x(0,t)$ and $g_2(t) = q_{xx}(0,t)$, just for ease of notation; these will be eliminated later.

Here $w(k) = -ik^3$, so the formula (28) yields:

$$Q = \Big(\frac{k^3 - l^3}{k - l}\Big)\Big|_{l=-i\frac{\partial}{\partial x}}(q) = -q_{xx} - ikq_x + k^2 q.$$

Thus the global relation (29) is:

$$e^{-ik^3 t}\hat{q}(k,t) = \hat{q}_0(k) - k^2\tilde{g}_0(-ik^3,t) + ik\tilde{g}_1(-ik^3,t) + \tilde{g}_2(-ik^3,t).$$

And the integral formula (30) is:

$$\begin{aligned} q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} & e^{ikx+ik^3 t}\hat{q}_0(k)\,\mathrm{d}k \\ & - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx+ik^3 t}\big(k^2\tilde{g}_0(-ik^3,t) - ik\tilde{g}_1(-ik^3,t) - \tilde{g}_2(-ik^3,t)\big)\,\mathrm{d}k, \end{aligned}$$

where the domain D^+ in this case is the infinite sector $\{k \in \mathbb{C} : \frac{\pi}{3} < \arg(k) < \frac{2\pi}{3}\}$.

To cancel the unwanted boundary conditions, we need transformations ν such that $\nu(k)^3 = k^3$. Using $\nu(k) = \omega k$ and $\nu(k) = \omega^2 k$, where ω is the cube root of unity, transforms the global relation to the following equations valid for $k \in \partial D^+$:

$$\begin{aligned} e^{-ik^3 t}\hat{q}(\omega k,t) &= \hat{q}_0(\omega k) - \omega^2 k^2\tilde{g}_0 + i\omega k\tilde{g}_1 + \tilde{g}_2 \\ e^{-ik^3 t}\hat{q}(\omega^2 k,t) &= \hat{q}_0(\omega^2 k) - \omega k^2\tilde{g}_0 + i\omega^2 k\tilde{g}_1 + \tilde{g}_2(-ik^3,t) \end{aligned}$$

A little algebraic manipulation yields the final result, a formula for q which depends only on q_0 and g_0 and not on g_1 and g_2:

$$\begin{aligned} q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} & e^{ikx+ik^3 t}\hat{q}_0(k)\,\mathrm{d}k \\ & - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx+ik^3 t}\big(3k^2\tilde{g}_0(-ik^3,t) - \omega\hat{q}_0(\omega k) - \omega^2\hat{q}_0(\omega^2 k)\big)\,\mathrm{d}k. \end{aligned}$$

□

A major advantage of the unified transform method is that, unlike traditional transform methods, it constructs representations of solutions which are always uniformly convergent at the boundaries of the domain. This makes it much more straightforward to verify that the solution does indeed satisfy the appropriate boundary value conditions [104]. Because the final substitution and elimination is so symmetrical in the different boundary conditions, the unified transform method is also well-suited to solving a wide variety of different boundary value problems: Dirichlet, Neumann, Robin, and more. Furthermore, the unified transform method lends itself well to numerical computations, more so than many classical transform methods. The numerical aspect has been much explored in the literature, see for example [16, 34, 60].

2.3.2 The setup for a linear fractional PDE

There exists some previous work analysing fractional PDEs on the half-line using methods similar to the unified transform method: the recent work of Arciga et al [13, 14, 128] and some papers of Kaikina [81, 82] both fall into this category. However, some of these have used other fractional models than the classical Riemann–Liouville one, e.g. Riesz or Abel fractional derivatives, while others have considered a narrower class of PDEs than that which we shall analyse here, or used more complicated methods than the unified transform method. There are also issues surrounding branch cuts, which naturally arise when non-integer power functions are introduced; some of the above-cited papers have skirted around these issues, while we address them head-on and consider what deformations of contours in the complex plane are permissible when branch cuts are excluded.

Our work here is rigorous and elementary, discussing and avoiding several potential pitfalls and ending up with a clear explicit formula for solutions to a large class of linear fractional PDEs on the half-line. Specifically, we need to address the problems which arise when a simple polynomial is replaced by a general linear combination of power functions, and we also need to consider and resolve the issue of branch cuts arising from the complex power functions involved in the analysis.

We consider the following general form of fractional linear PDE:

$$q_t + w\left(-i\frac{\partial}{\partial x}\right)q = 0, \qquad x \in (0,\infty), t \in (0,T), \tag{31}$$

where w is a finite fractional series of power functions. More explicitly, we write

$$w(k) = \sum_{\alpha} c_\alpha k^\alpha, \qquad \alpha, c_\alpha \in \mathbb{C}, \mathrm{Re}(\alpha) > 0, \tag{32}$$

where the summation is finite, i.e. the indices α are contained in some finite set of complex numbers in the right half plane. Thus we can write the PDE more explicitly as

$$\frac{\partial q}{\partial t}+\sum_{\alpha}c_{\alpha}(-i)^{\alpha}\frac{\partial^{\alpha}q}{\partial x^{\alpha}}=0,\qquad x\in(0,\infty),t\in(0,T). \tag{33}$$

We shall attempt to solve this PDE with the initial condition $q(x,0)=q_0(x)$, $x\in\mathbb{R}^+$, where the function $q_0:[0,\infty)\to\mathbb{C}$ is given and has a well-defined Fourier transform $\hat{q}_0$. We will also need boundary conditions, but their number and nature will be determined later.

For the purposes of this chapter, all fractional derivatives are defined in the Riemann–Liouville sense with the constant of differintegration being 0. The Riemann–Liouville model is the natural one to use because of how well it interacts with Fourier and Laplace transforms (see Lemmas 1.1.4 and 1.1.5), while 0 is a logical lower bound for integration because the spatial domain for the PDE is bounded below by 0. We are using the half-line $x\in[0,\infty)$ as our spatial domain because the equation on the full line would be relatively easy to solve using a standard Fourier transform method. For the problem on the half-line, an analytic solution of the PDE requires more advanced transform methods.

Since we shall be using fractional power functions, we must define domains and branches for these functions. We declare that all fractional power functions are defined using the principal branch with branch cut along the negative real axis, i.e.

$$k^{\alpha}=r^{\alpha}e^{i\theta\alpha},\qquad k=re^{i\theta},r\in\mathbb{R}^+,\theta\in(-\pi,\pi). \tag{34}$$

For the avoidance of doubt, let us write explicitly the precise formula we are using for Fourier transforms with respect to x, namely a half-Fourier transform defined on $[0,\infty)$ and denoted by $\hat{}$:

$$\hat{q}(k,t)=\int_0^{\infty}e^{-ikx}q(x,t)\,\mathrm{d}x. \tag{35}$$

We do not specify here a particular transform with respect to t, because which type of transform we use for this will depend on the approach taken, and it is often more complicated than a simple Fourier transform like (35).

We also state here the following fractional generalisation of the integration by parts law, whose proof can be found in [7], and which we shall need to use in §2.3.4 below.

Lemma 2.3.2 (Integration by parts)**.** *Let $[a,b]$ be an interval in $\mathbb{R}$ and α be a complex number with* $\mathrm{Re}(\alpha)>0$. *We have*

$$\int_a^b f(x)\cdot{}_aD_x^{\alpha}g(x)\,\mathrm{d}x=\int_a^b g(x)\cdot{}_x^{C}D_b^{\alpha}f(x)\,\mathrm{d}x-\sum_{j=0}^{n-1}\Big[(-1)^{n+j}\,{}_aD_x^{\alpha-n+j}g(x)\cdot{}_aD_x^{n-j-1}f(x)\Big]_a^b, \tag{36}$$

provided the relevant differintegrals exist, where $n:=\lceil\mathrm{Re}(\alpha)\rceil$.

2.3.3 Finding the global relation using a divergence form

Here we work through the first two steps of the method as laid out in §2.3.1, writing the PDE (31) in divergence form and deriving a global relation. The biggest challenge here, as we shall see, is to define the function $Q(x,t,k)$ in an appropriate way so that the rest of the argument works.

The fractional PDE (31) can still be written in divergence form as (27), provided that the function $Q(x,t,k)$ satisfies the following condition:

$$\big({}_0D_x - ik\big)Q = \big(w(k) - w(-i\,{}_0D_x)\big)q. \tag{37}$$

However, now that w is no longer necessarily a polynomial, the simple expression (28) for Q no longer applies. Already the presence of fractional derivatives makes the problem harder than in the classical case. How do we now find an explicit form for Q?

Intuitively, we can still consider the function $\frac{w(k)-w(l)}{k-l}$ with the idea of setting $l = -i\,{}_0D_x$ at some later stage. One idea would be to expand $(k-l)^{-1}$ as a power series, namely to write it as

$$(k-l)^{-1} = \begin{cases} -l^{-1}(1+kl^{-1}+k^2l^{-2}+k^3l^{-3}+\dots) = \sum\limits_{j=0}^{\infty} -k^jl^{-j-1}, & |k|<1; \\ \\ k^{-1}(1+k^{-1}l+k^{-2}l^2+k^{-3}l^3+\dots) = \sum\limits_{j=0}^{\infty} k^{-j-1}l^j, & |k|>1. \end{cases}$$

Multiplying this series by $w(k)-w(l)$ would yield an expression for $\frac{w(k)-w(l)}{k-l}$ as an infinite series of terms of the form $k^\alpha l^\beta$, namely:

$$i(k-l)^{-1}\left(w(k)-w(l)\right) = \begin{cases} i\sum\limits_{j=0}^{\infty}\sum\limits_{\alpha} c_\alpha\left[-k^{\alpha+j}l^{-j-1}+k^jl^{\alpha-j-1}\right], & |k|<1; \\ \\ i\sum\limits_{j=0}^{\infty}\sum\limits_{\alpha} c_\alpha\left[k^{\alpha-j-1}l^j-k^{-j-1}l^{\alpha+j}\right], & |k|>1. \end{cases}$$

The catch is that fractional differential operators do not have a semigroup property, by Lemma 1.1.7: after setting $l=-i\,{}_0D_x$, the product of l^a and l^b will not necessarily be l^{a+b}. So the above manipulation of terms is actually not valid if $l=-i\,{}_0D_x$ is assumed *a priori*.

However, we do not need Q to be precisely the expression $i\left(\frac{w(k)-w(l)}{k-l}\right)$ with l replaced by $-i\,{}_0D_x$. Any function Q that satisfies the condition (37) will automatically give the divergence form (27) as an equivalent formulation of the PDE (31). So all we really need to do is to find a function Q satisfying (37); considering $\frac{w(k)-w(l)}{k-l}$ may be useful as motivation to tell us where to look for such a function, but no more than that. With this in mind, let us try defining Q by the above series with $l = -i\,{}_0D_x$, ignoring whether our manipulations of l would actually be valid for this differential operator. The resulting expression for Q is:

$$Q(x,t,k) = \begin{cases} i\sum_{j=0}^{\infty}\sum_{\alpha} c_\alpha \left[k^j\left(-i\,{}_0D_x\right)^{\alpha-j-1}(q) - k^{\alpha+j}\left(-i\,{}_0D_x\right)^{-j-1}(q)\right], & |k|<1; \\ i\sum_{j=0}^{\infty}\sum_{\alpha} c_\alpha \left[k^{\alpha-j-1}\left(-i\,{}_0D_x\right)^{j}(q) - k^{-j-1}\left(-i\,{}_0D_x\right)^{\alpha+j}(q)\right] & \\ \qquad\qquad\qquad\qquad\qquad\qquad +A(k)e^{ikx}, & |k|>1; \end{cases} \tag{38}$$

where $A(k)$ is chosen so that Q is continuous across $|k| = 1$. Such a function $A(k)$ exists because $A(k)e^{ikx}$ is the general solution of the ODE $\left({}_0D_x - ik\right)Q = 0$ in x.

It is now straightforward to verify, using the fact that ${}_0D_x \circ {}_0D_x^{\alpha} = {}_0D_x^{\alpha+1}$ for all $\alpha \in \mathbb{C}$ by Lemma 1.1.7, that both of the above series expressions for $Q(x,t,k)$ do satisfy (37) as required. So (38) is a valid possibility for the function Q.

The problem now is that an infinite series expression for Q is unwieldy and difficult to deal with. It looks as though integrating the divergence form (27) and using the expressions (38) for Q will lead to a global relation in the form of an infinite series, with infinitely many boundary terms involved, and therefore requiring infinitely many boundary conditions specified in order to have a unique solution. This would be disastrous, since it is already known (see e.g. [82]) that only finitely many boundary conditions need to be specified in order to get a unique solution for a PDE of this form.

But in fact it turns out that almost all the terms in the infinite series for Q cancel out when substituted into the global relation!

The next step in the method as laid out in §2.3.1 is to apply integrals with respect to both x and t to the divergence form (27) of the PDE. This yields the following relation between integral transforms:

$$e^{w(k)t}\hat{q}(k,t) - \hat{q}_0(k) = -\int_0^t e^{w(k)\tau} Q(0,\tau,k)\,d\tau, \qquad \mathrm{Im}(k) < 0. \tag{39}$$

Here we have assumed sufficient decay conditions that the upper limit term for x vanishes, i.e.

$$\lim_{x\to\infty}\int_0^t e^{-ikx+w(k)\tau} Q(x,\tau,k)\,d\tau = 0. \tag{40}$$

Importantly, the only time Q appears in (39) is when $x = 0$: we do not need to deal with the full complexity of the function $Q(x,t,k)$, but only with the special case $Q(0,\tau,k)$. And by definition of the Riemann–Liouville fractional integral, the function ${}_0D_x^{\nu} f(x)\big|_{x=0}$ for any given f is always identically zero when this is a fractional integral, i.e. when $\mathrm{Re}(\nu) < 0$. So for the purposes of the global relation (39), we can ignore all terms in the infinite series of (38) in which ${}_0D_x$ appears to a negative power.

We consider the $|k| < 1$ part of (38), since negative powers of ${}_0D_x$ appear in only finitely many terms of the $|k| > 1$ part and in all but finitely many terms of the $|k| < 1$ part. Substituting $x = 0$ into the $|k| < 1$ series from (38), we find:

$$\begin{aligned} Q(0,\tau,k) &= i\sum_{j=0}^{\infty}\sum_{\alpha} c_\alpha k^j \left(-i\,{}_0D_x\right)^{\alpha-j-1} q(0,\tau) \\ &= -\sum_{\alpha} c_\alpha(-i)^\alpha \sum_{j=0}^{\lfloor \mathrm{Re}(\alpha)\rfloor-1} (ik)^j\,{}_0D_x^{\alpha-j-1} q(0,\tau), \qquad |k| < 1. \end{aligned}$$

Thus the identity (39) yields the following global relation:

$$e^{w(k)t}\hat{q}(k,t) - \hat{q}_0(k) = \sum_{\alpha} c_\alpha(-i)^\alpha \sum_{j=0}^{\lfloor \mathrm{Re}(\alpha)\rfloor-1} (ik)^j \int_0^t e^{w(k)\tau}\,{}_0D_x^{\alpha-j-1} q(0,\tau)d\tau, \tag{41}$$

valid for $\mathrm{Im}(k) < 0$ and $|k| < 1$. However, now that the infinite series over j has become a finite one, we no longer need $|k| < 1$ for convergence. So by analytic continuation, (41) is valid for all k in the lower half plane $\mathrm{Im}(k) < 0$. Thus we have a finite closed-form global relation as desired.

2.3.4 Finding the global relation using double transforms

Here we consider another way of deriving the global relation. This does not follow the approach which was indicated in §2.3.1 for non-fractional PDEs, but it is similar to a known alternative methodology [128], and we shall see that it is equivalent in terms of the final identity obtained.

Before proceeding to analyse the PDE, we first obtain an identity which we shall need to use below. Applying the fractional integration by parts rule (36) to the functions $f(x) = e^{-ikx}$ and $g(x) = q(x,t)$, with upper and lower limits $a = 0$ and $b \to \infty$, and using the formula (4) for fractional differintegrals of exponential functions, we find

$$\begin{aligned} &\int_0^\infty e^{-ikx}\cdot{}_0D_x^\alpha q(x,t)\,\mathrm{d}x \\ &\quad= \int_0^\infty q(x,t)\cdot(ik)^\alpha e^{-ikx}\,\mathrm{d}x - \sum_{j=0}^{n-1}\left[(-1)^{n+j}\,{}_0D_x^{\alpha-n+j}q(x,t)\cdot(-ik)^{n-j-1}e^{-ikx}\right]_0^\infty, \end{aligned}$$

or in other words

$$\widehat{\frac{\partial^\alpha q}{\partial x^\alpha}}(k,t) = (ik)^\alpha\hat{q}(k,t) + \sum_{j=0}^{n-1}(ik)^{n-j-1}\left[e^{-ikx}\,{}_0D_x^{\alpha-n+j}q(x,t)\right]_0^\infty. \tag{42}$$

We also use ˜ to denote the Laplace transform with respect to t:

$$\tilde{q}(x,s) = \int_0^\infty e^{-st} q(x,t)\,\mathrm{d}t. \tag{43}$$

For the purposes of the definition (43), we extend the function q beyond the interval $[0,T]$ by making it identically zero for large t. This will have no effect on the final result, because the only appearance of Laplace transforms will be to be applied and then almost immediately inverted again.

Armed with the integration by parts identity (42), we proceed to apply a half-Fourier transform with respect to x to the PDE (33):

$$\begin{aligned}
(33) \Rightarrow \frac{\partial \hat{q}}{\partial t} + \sum_\alpha c_\alpha (-i)^\alpha \widehat{\frac{\partial^\alpha q}{\partial x^\alpha}} &= 0 \\
\Rightarrow \frac{\partial \hat{q}}{\partial t} + \sum_\alpha c_\alpha (-i)^\alpha \left[(ik)^\alpha \hat{q}(k,t) + \sum_{j=0}^{n-1} (ik)^{n-j-1} \left[e^{-ikx}\, {}_0D_x^{\alpha-n+j} q(x,t) \right]_0^\infty \right] &= 0 \\
\Rightarrow \frac{\partial \hat{q}}{\partial t} + w(k)\hat{q} + \sum_\alpha c_\alpha (-i)^\alpha \sum_{j=0}^{n-1} (ik)^{n-j-1} \left[e^{-ikx}\, {}_0D_x^{\alpha-n+j} q(x,t) \right]_0^\infty &= 0.
\end{aligned}$$

Note that the notation $n = \lceil \mathrm{Re}(\alpha) \rceil$ is propagated here from Lemma 2.3.2.

Next, we apply a Laplace transform with respect to t. This results in an expression for the double-transformed function $\tilde{\hat{q}}(k,s)$:

$$s\tilde{\hat{q}}(k,s) - \hat{q}(k,0) + w(k)\tilde{\hat{q}}(k,s) + \sum_\alpha c_\alpha (-i)^\alpha \sum_{j=0}^{n-1} (ik)^{n-j-1} \left[e^{-ikx}\, {}_0D_x^{\alpha-n+j} \tilde{q}(x,s) \right]_0^\infty = 0,$$

which rearranges to

$$\tilde{\hat{q}}(k,s) = \frac{\hat{q}(k,0) + \sum_\alpha c_\alpha (-i)^\alpha \sum_{j=0}^{n-1} (ik)^{n-j-1} \left[e^{-ikx}\, {}_0D_x^{\alpha-n+j} \tilde{q}(x,s) \right]_0^\infty}{s + w(k)}.$$

Now we have an explicit formula for a transform of q in terms of some initial and boundary conditions. We apply an inverse Laplace transform with respect to t, recalling both the convolution theorem and the fact that the transform of $e^{-w(k)t}$ is $\frac{1}{s+w(k)}$:

$$\begin{aligned}\hat{q}(k,t) &= \hat{q}(k,0)e^{-w(k)t} \\ &\qquad - \left[\sum_{\alpha} c_\alpha(-i)^\alpha \sum_{j=0}^{n-1}(ik)^{n-j-1}\left[e^{-ikx}\,{}_0D_x^{\alpha-n+j}q(x,t)\right]_0^\infty\right] * \left[e^{-w(k)t}\right] \\ &= \hat{q}(k,0)e^{-w(k)t} \\ &\qquad - \int_0^t \sum_{\alpha} c_\alpha(-i)^\alpha \sum_{j=0}^{n-1}(ik)^{n-j-1}\left[e^{-ikx}\,{}_0D_x^{\alpha-n+j}q(x,\tau)\right]_0^\infty e^{-w(k)(t-\tau)}\,\mathrm{d}\tau.\end{aligned}$$

Thus the global relation is

$$e^{w(k)t}\hat{q}(k,t) = \hat{q}(k,0) - \sum_{\alpha} c_\alpha(-i)^\alpha \sum_{j=0}^{n-1}(ik)^{n-j-1}\left[e^{-ikx}\int_0^t e^{w(k)\tau}\,{}_0D_x^{\alpha-n+j}q(x,\tau)\,\mathrm{d}\tau\right]_0^\infty. \tag{44}$$

It is valid for $\mathrm{Im}(k) < 0$, this condition being required for the half-Fourier transforms with respect to x to be well-defined.

For ease of notation, we define

$$g(k,t) := \sum_{\alpha} c_\alpha(-i)^\alpha \sum_{j=0}^{n-1}(ik)^{n-j-1}\left[e^{-ikx}\int_0^t e^{w(k)\tau}\,{}_0D_x^{\alpha-n+j}q(x,\tau)\,\mathrm{d}\tau\right]_0^\infty, \tag{45}$$

so that the global relation is

$$e^{w(k)t}\hat{q}(k,t) = \hat{q}(k,0) - g(k,t), \qquad \mathrm{Im}(k) < 0. \tag{46}$$

We note that, as expected, the global relation (44) is identical, under the assumption (40) on the decay of q at infinity, to the previously obtained global relation (41). The discrepancy in the number of terms in the series (namely, $\lfloor\mathrm{Re}(\alpha)\rfloor$ in (41) versus $\lceil\mathrm{Re}(\alpha)\rceil$ in (44)) is resolved by using the same argument as in §2.3.3 to point out that the fractional integral ${}_0D_x^{\alpha-\lceil\mathrm{Re}(\alpha)\rceil}q(0,t)$ is identically zero for any α with non-integer real part.

This acts as confirmation that our method is working: we have obtained exactly the same identity twice using two different approaches.

2.3.5 Deducing the solution

We start from the global relation (46) and apply an inverse half-Fourier transform with respect to x. This yields the following, our first explicit expression for q:

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-w(k)t}\left(\hat{q}(k,0) - g(k,t)\right)\mathrm{d}k. \tag{47}$$

Note that the function $e^{ikx-w(k)t}g(k,t)$ is analytic in k everywhere except along the branch cut for $w(k)$; so by our definition (34), it is analytic on the domain $\mathbb{C}\backslash(-\infty,0]$ for k. Furthermore it has exponential decay (tends to zero) as $|k|\to\infty$ with $\mathrm{Im}(k)>0, \mathrm{Re}(w(k))>0$. So by Cauchy's theorem, we can deform the contour of integration for the second half of the integral in (47) through any region with $\mathrm{Im}(k)>0$ and $\mathrm{Re}(w(k))>0$. Thus, we obtain the following improved explicit formula for q:

$$q(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ikx-w(k)t}\hat{q}(k,0)\,\mathrm{d}k-\frac{1}{2\pi}\int_{\partial D^+}e^{ikx-w(k)t}g(k,t)\,\mathrm{d}k, \tag{48}$$

where the domain D^+ is defined as

$$D^+=\{k\in\mathbb{C}:\mathrm{Im}(k)>0,\mathrm{Re}(w(k))<0\} \tag{49}$$

and we assume that $\mathrm{Re}(w(k))\geq 0$ for all $k\in\mathbb{R}$.

The equation (48) gives us an expression for q in terms of a single initial condition, namely $q(x,0)$, and n boundary conditions, namely $D_x^{\alpha-n+j}q(0,t)$ for $0\leq j<n-1$, provided we have sufficient decay conditions on q as $x\to\infty$. But this is an overdetermined problem, since we do not need that many boundary conditions [81, 82].

Fortunately, it is possible to eliminate some of the boundary conditions by using the global relation (44) with k replaced by $\nu(k)$ for some function ν. This function is required to satisfy two properties:

- It must preserve w, i.e. $w(\nu(k))=w(k)$. This is so that all the terms involving $w(k)$ in (44) are not altered by the substitution, while those directly involving k may change.
- It must map the boundary ∂D^+ into some region of the lower half k-plane. This is so that the global relation is valid at $\nu(k)$ when $k\in\partial D^+$.

Such functions ν are hard to find explicitly for the most general function w. But given a specific w, it is often possible to find the ν required. For example, in the simple case of $w(k)=k^\alpha$, we can take $\nu(k)=e^{2\pi im/\alpha}$ for any integer value of m.

Using the functions ν, we find new equations in a similar form to (44) which are valid for $k\in\partial D^+$ and therefore can be substituted into (48). By making multiple such substitutions, it is possible to eliminate half of the boundary conditions, leaving only half that need to be specified in the initial setup of the problem.

2.3.6 Summary and verification

We conclude the following result.

Theorem 2.3.3. *Given a PDE of the form* (31) *valid on the region* $0 < x < \infty, 0 < t < T$*, where* w *is a finite series of power functions defined by* (32) *and satisfying* $\mathrm{Re}(w(k)) \geq 0$ *for all* $k \in \mathbb{R}$*, and fractional derivatives are defined in the Riemann–Liouville sense with constant of differintegration* 0*, the unified transform method can be used to construct an explicit solution* $q(x,t)$ *in terms of the initial condition* $q(x,0)$ *and some boundary conditions* $\frac{\partial^{\alpha-r}q}{\partial x^{\alpha-r}}(0,t)$.

In order to verify this result, we substitute the formula (48) into the original PDE (31) to check that this q does indeed satisfy the equation. (The final formula would be in a more complicated form than (48), but from that formula it is easy to return to the expression (48) just by reversing the substitutions made to get there – all modifications between the two are only a matter of rewriting boundary conditions in terms of each other.)

Starting from the formula (48) for q, we find

$$\frac{\partial q}{\partial t} = -\frac{1}{2\pi}\int_{-\infty}^{\infty} w(k)e^{ikx-w(k)t}\hat{q}(k,0)\,\mathrm{d}k + \frac{1}{2\pi}\int_{\partial D^+} w(k)e^{ikx-w(k)t}g(k,t)\,\mathrm{d}k - \frac{1}{2\pi}\int_{\partial D^+} e^{ikx-w(k)t}g_t(k,t)\,\mathrm{d}k,$$

and, using (42),

$$w\left(-i\frac{\partial}{\partial x}\right)q = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\Bigg[w(k)\hat{q}(k,t) + \sum_{\alpha} c_\alpha(-i)^\alpha \sum_{j=0}^{n-1}(ik)^{n-j-1}\left[e^{-ikx}\,{}_0D_x^{\alpha-n+j}q(x,t)\right]_0^\infty\Bigg]\,\mathrm{d}k.$$

So the left-hand side of the original PDE evaluates as follows:

$$\begin{aligned}
&\frac{\partial q}{\partial t} + w\left(-i\frac{\partial}{\partial x}\right)q \\
&\quad = -\frac{1}{2\pi}\int_{-\infty}^{\infty} w(k)e^{ikx-w(k)t}\hat{q}(k,0)\,\mathrm{d}k + \frac{1}{2\pi}\int_{\partial D^+} w(k)e^{ikx-w(k)t}g(k,t)\,\mathrm{d}k \\
&\quad\quad -\frac{1}{2\pi}\int_{\partial D^+} e^{ikx-w(k)t}\sum_{\alpha} c_\alpha(-i)^\alpha\sum_{j=0}^{n-1}(ik)^{n-j-1}\left[e^{-ikx+w(k)t}\,{}_0D_x^{\alpha-n+j}q(x,t)\right]_0^\infty\,\mathrm{d}k \\
&\quad\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty} w(k)e^{ikx}\left[e^{-w(k)t}\hat{q}(k,0) - e^{-w(k)t}g(k,t)\right]\mathrm{d}k \\
&\quad\quad + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\sum_{\alpha} c_\alpha(-i)^\alpha\sum_{j=0}^{n-1}(ik)^{n-j-1}\left[e^{-ikx}\,{}_0D_x^{\alpha-n+j}q(x,t)\right]_0^\infty\,\mathrm{d}k,
\end{aligned}$$

which is zero since Cauchy's theorem enables us to equate $\int_{\partial D^+}$ with $\int_{-\infty}^{\infty}$ when required. We also used the global relation (44) as a substitution for $\hat{q}(k,t)$ in the derivation of the above formula.

Thus we have proved that the solution constructed above actually is a solution, subject to a straightforward verification of the initial and boundary value conditions.

2.3.7 Examples and extensions

As a basic but important example of the method outlined in general above, let us consider the case where w is a single power function, say

$$w(k) = -A(ik)^\alpha,$$

where the constants A and α are fixed. For simplicity we now assume both of these constants to be positive real.

In other words, we shall attempt to solve the following PDE:

$$\frac{\partial q}{\partial t} = A\frac{\partial^\alpha q}{\partial x^\alpha}, \qquad x \in (0,\infty), t \in (0,T). \tag{50}$$

As always, we impose a single initial condition,

$$q(x,0) = q_0(x), \qquad x \in (0,\infty),$$

and a certain number of boundary conditions to be determined later.

In this case, the global relation (41) is:

$$e^{-A(ik)^\alpha t}\hat{q}(k,t) = \hat{q}_0(k) - A\sum_{j=0}^{\lfloor\alpha\rfloor-1}(ik)^j\int_0^t e^{-A(ik)^\alpha\tau}\,{}_0D_x^{\alpha-j-1}q(0,\tau)d\tau, \qquad \mathrm{Im}(k) < 0. \tag{51}$$

In order to find the region D^+ defined by (49), we note that the following conditions are equivalent:

$$\begin{aligned}
&\mathrm{Re}(w(k)) < 0;\\
&\mathrm{Re}\left((ik)^\alpha\right) > 0;\\
2n\pi - \frac{\pi}{2} < \arg&\left((ik)^\alpha\right) < 2n\pi - \frac{\pi}{2} \qquad \text{for some } n \in \mathbb{Z};\\
\frac{\pi}{\alpha}\left(2n - \frac{1}{2}\right) < \arg&(ik) < \frac{\pi}{\alpha}\left(2n + \frac{1}{2}\right) \qquad \text{for some } n \in \mathbb{Z}.
\end{aligned}$$

We saw in §2.3.5 that all real k are required *not* to satisfy this condition, in order that a meaningful deformation of contours can be applied. Thus we require that

$$\left|\frac{2n\pi}{\alpha} \pm \frac{\pi}{2}\right| \geq \frac{\pi}{2\alpha}$$

for all integers n, i.e. that $|4n \pm \alpha| \geq 1$ for all integers n. In other words, α must lie in one of the intervals $[1,3]$, $[5,7]$, $[9,11]$, etc.

The domain D^+ can be described as follows, working from (49):

$$\begin{aligned}
D^+ &= \{k \in \mathbb{C} : \mathrm{Im}(k) > 0, \mathrm{Re}\left((ik)^\alpha\right) > 0\} \\
&= \left\{k \in \mathbb{C} : 0 < \arg(k) < \tfrac{\pi}{2}, 2n\pi - \tfrac{\pi}{2} < \alpha\left[\arg(k) + \tfrac{\pi}{2}\right] < 2n\pi + \tfrac{\pi}{2}, n \in \mathbb{Z}\right\} \\
&\qquad \cup \left\{k \in \mathbb{C} : \tfrac{\pi}{2} < \arg(k) < \pi, 2n\pi - \tfrac{\pi}{2} < \alpha\left[\arg(k) - \tfrac{3\pi}{2}\right] < 2n\pi + \tfrac{\pi}{2}, n \in \mathbb{Z}\right\} \\
&= \left\{k \in \mathbb{C} : \alpha\left[\arg(k) + \tfrac{\pi}{2}\right] \in \left(\tfrac{\pi\alpha}{2}, \pi\alpha\right) \cap \left(2n\pi - \tfrac{\pi}{2}, 2n\pi + \tfrac{\pi}{2}\right), n \in \mathbb{Z}\right\} \\
&\qquad \cup \left\{k \in \mathbb{C} : \alpha\left[\arg(k) - \tfrac{3\pi}{2}\right] \in \left(-\pi\alpha, -\tfrac{\pi\alpha}{2}\right) \cap \left(2n\pi - \tfrac{\pi}{2}, 2n\pi + \tfrac{\pi}{2}\right), n \in \mathbb{Z}\right\}.
\end{aligned}$$

In particular, if $1 < \alpha \leq \frac{3}{2}$, then the intervals on the right do not intersect and so D^+ is empty. For now, let us assume $\boldsymbol{\frac{3}{2} < \alpha < \frac{5}{2}}$, so that we have:

$$\begin{aligned}
D^+ &= \left\{k \in \mathbb{C} : \alpha\left[\arg(k) + \tfrac{\pi}{2}\right] \in \left(\tfrac{\pi\alpha}{2}, \pi\alpha\right) \cap \left(\tfrac{3\pi}{2}, \tfrac{5\pi}{2}\right)\right\} \\
&\qquad \cup \left\{k \in \mathbb{C} : \alpha\left[\arg(k) - \tfrac{3\pi}{2}\right] \in \left(-\pi\alpha, -\tfrac{\pi\alpha}{2}\right) \cap \left(-\tfrac{5\pi}{2}, -\tfrac{3\pi}{2}\right)\right\} \\
&= \left\{k \in \mathbb{C} : \tfrac{3\pi}{2\alpha} - \tfrac{\pi}{2} < \arg(k) < -\tfrac{3\pi}{2\alpha} + \tfrac{3\pi}{2}\right\}.
\end{aligned} \tag{52}$$

The integral expression (48) for q now becomes:

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx + A(ik)^\alpha t}\hat{q}_0(k)\,\mathrm{d}k - \frac{1}{2\pi}\int_\Gamma e^{ikx + A(ik)^\alpha t} g(k,t)\,\mathrm{d}k, \tag{53}$$

with notation defined as follows. The contour Γ runs along the boundary of D^+, forming a V-shape in the upper half plane:

$$\Gamma = \left\{re^{(3\alpha-3)\pi i/2\alpha} : \infty > r > 0\right\} \cup \left\{re^{(3-\alpha)\pi i/2\alpha} : 0 < r < \infty\right\}. \tag{54}$$

And the function $g(k,t)$ is defined by minus the main term on the right-hand side of (51), this being the notation required for the global relation to be expressible in the form (46). In other words, g is given by the following expression:

$$g(k,t) = A\sum_{j=0}^{\lfloor\alpha\rfloor - 1}(ik)^j\int_0^t e^{-A(ik)^\alpha\tau}\,{}_0D_x^{\alpha-j-1}(q)d\tau. \tag{55}$$

Now we need to find functions ν which preserve the power function w, i.e. such that

$$(\nu(k))^\alpha = k^\alpha.$$

This is easy to solve for ν; the function

$$\nu(k) = e^{2n\pi i/\alpha}k$$

will work for any integer n such that $\arg(k)$ and $\frac{2n\pi}{\alpha} + \arg(k)$ are both in the domain $(-\pi, \pi)$ required by the power function with branch cut along the negative real axis. We also require $\frac{2n\pi}{\alpha} + \arg(k)$ to be in the domain $(-\pi, 0)$ when k is on the contour Γ, in order that our substitution into the global relation will be valid.

Clearly any positive n would not send the contour Γ into the lower half plane. Furthermore, any $n \leq -2$ would send Γ outside of the domain $(-\pi, \pi)$ for arguments. The case $n = 0$ only gives us the identity map. So the only non-trivial possibility for ν is with $n = -1$, namely:

$$\nu(k) = e^{-2\pi i/\alpha}k. \tag{56}$$

This function ν sends Γ into the lower half plane if and only if $\alpha > \frac{7}{3}$. Let us now assume $\boldsymbol{2 < \alpha < \frac{7}{3}}$, so that we have both a valid map ν and a fixed value of $\lfloor\alpha\rfloor$. Now, substituing k for $\nu(k)$ into the global relation (51), we find:

$$\begin{aligned} & e^{-A(ik)^\alpha t}\hat{q}\left(e^{-2\pi i/\alpha}k, t\right) \\ & \quad = \hat{q}_0\left(e^{-2\pi i/\alpha}k\right) - A\sum_{j=0}^{1}\left(ie^{-2\pi i/\alpha}k\right)^j \int_0^t e^{-A(ik)^\alpha \tau}\, {}_0D_x^{\alpha-j-1}q(0,\tau)d\tau, \qquad k \in \Gamma. \end{aligned} \tag{57}$$

Substituting (57) into (53) enables us to eliminate one of the $\lfloor\alpha\rfloor = 2$ boundary conditions on the right-hand side, leaving only one boundary condition that needs to be specified in the initial setup of the problem.

Thus, in the case $2 < \alpha < \frac{7}{3}$, the unified transform method can be used to solve the fractional PDE (50), with the initial condition $q(x, 0) = q_0(x)$ and exactly one of the two boundary terms ${}_0D_x^{\alpha-1}q(0, t)$, ${}_0D_x^{\alpha-2}q(0, t)$ specified.

Of course, this range of values of α is not the only one in which the problem (50) can be solved. We chose our restrictions on α merely for convenience. It would be just as easy to solve the PDE in the case $\frac{5}{2} < \alpha < 3$, or other higher ranges of α. What we have presented here is the solution of a model problem, in order to demonstrate the methodology. Other example problems would work out similarly, but might become more complicated according to the value of α.

In all the analysis of (50), we assumed that the index α was real. But even this simple PDE becomes more interesting to solve when α is complex. The boundary of the domain D^+ would no longer consist of rays from the origin in the complex plane; instead, it would consist of infinite logarithmic spirals, due to considering the argument of k^α with α complex, and the contour of integration would become correspondingly more complicated.

There are also many ways in which the method laid out in §2.3.2 could be generalised beyond even the general equation (31).

For integer-order PDEs, the unified transform method has been applied to many families of equations more advanced than (26) on more complicated domains than the half-line $[0, \infty)$ – for example, finite intervals in the real line, convex polygons in a plane, and beyond [55]. Fractional analogues of these problems could be considered and potentially solved by modifying the unified transform method for the new scenario.

In this chapter we have considered only fractional differential equations of Riemann–Liouville type. But many real-world processes can be better modelled using other definitions of fractional calculus, such as the Caputo, Caputo–Fabrizio, and Atangana–Baleanu definitions. Solving fractional PDEs in these alternative fractional models could be an important result, and it may be possible to do so using the unified transform method. The Riemann–Liouville model has the unique advantage of interacting with Fourier and Laplace transforms in the way one would expect (Lemmas 1.1.4 and 1.1.5), but other models have similar properties with the power function replaced by more complicated multipliers, and PDEs in these models are still amenable to transform approaches.

2.4 A study of the Atangana–Baleanu model

2.4.1 Introduction

In the **Atangana–Baleanu** (or **AB**) model of fractional calculus, proposed in 2016 by Atangana and Baleanu [18], the power function appearing as the kernel of the Riemann–Liouville fractional integral (1) is replaced by a Mittag-Leffler function. As discussed earlier in §1.1.2, the motivation for this definition was to describe non-local dynamics in a new way, using the Mittag-Leffler function which is already known to be highly useful in fractional calculus.

Fractional derivatives in the AB model are defined by the following alternative formulae, of Riemann–Liouville and Caputo type respectively:

$$\begin{aligned} {}^{ABR}_{a}D^{\alpha}_{t}f(t) &= \tfrac{B(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t f(x)E_{\alpha}\Big(\tfrac{-\alpha}{1-\alpha}(t-x)^{\alpha}\Big)\,\mathrm{d}x, \\ {}^{ABC}_{a}D^{\alpha}_{t}f(t) &= \tfrac{B(\alpha)}{1-\alpha}\int_a^t f'(x)E_{\alpha}\Big(\tfrac{-\alpha}{1-\alpha}(t-x)^{\alpha}\Big)\,\mathrm{d}x, \end{aligned}$$

Fractional integrals in the AB model are defined by

$${}^{AB}_{a}I^{\alpha}_{t}f(t) = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)}\,{}^{RL}_{a}I^{\alpha}_{t}f(t), \tag{58}$$

where ${}^{RL}_{a}I^{\alpha}$ is the Riemann–Liouville fractional integral. In all three cases, the definitions are valid for $0 < \alpha < 1$, and $B(\alpha)$ is a normalisation function. We shall formalise these definitions and examine the required assumptions more carefully in §2.4.2 below.

Applications of the AB model have been explored in fields as diverse as chaos theory [8], heat transfer [18], and variational problems [1]. It has also been considered from a numerical viewpoint in [37]. Some mathematical properties of AB differintegrals had already been proven before my work in this area: for example, the original paper [18] established the formulae for Laplace transforms of AB differintegrals and some Lipschitz-type inequalities; the paper [1] considered integration by parts identities and Euler-Lagrange equations; and the paper [37] established, using Laplace transforms, analogues of the Newton–Leibniz formula for the integral of a derivative.

This chapter represents one of the major early contributions to the theory of the AB fractional model: providing firm theoretical groundwork for this model, solving new problems in the AB framework, and considering how certain ideas from classical fractional calculus can be extended here. In particular, one of the major results of this work is Theorem 2.4.4, establishing a new series formula for AB fractional derivatives which, as well as enabling the proof of new theorems and being potentially useful in numerical applications, links the AB model back to the classical Riemann–Liouville model.

2.4.2 Rigorous formulation and a series formula

We formulate the definitions of AB derivatives, of Riemann–Liouville and Caputo type, rigorously as follows.

Definition 2.4.1. The ABR fractional derivative (R denotes Riemann–Liouville type) is defined by

$$ {}^{ABR}_{a}D^{\alpha}_{t}f(t)=\tfrac{B(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_{a}^{t}f(x)E_{\alpha}\left(\tfrac{-\alpha}{1-\alpha}(t-x)^{\alpha}\right)\mathrm{d}x, \tag{59}$$

for $0<\alpha<1$, $a<t<b$, and $f\in L^{1}(a,b)$.

Definition 2.4.2. The ABC fractional derivative (C denotes Caputo type) is defined by

$$ {}^{ABC}_{a}D^{\alpha}_{t}f(t)=\tfrac{B(\alpha)}{1-\alpha}\int_{a}^{t}f'(x)E_{\alpha}\left(\tfrac{-\alpha}{1-\alpha}(t-x)^{\alpha}\right)\mathrm{d}x, \tag{60}$$

for $0<\alpha<1$, $a<t<b$, and f a differentiable function on $[a,b]$ such that $f'\in L^{1}(a,b)$.

In the above definitions, the function E_{α} is the Mittag-Leffler function, defined by:

$$E_{\alpha}(x)=\sum_{n=0}^{\infty}\frac{x^{n}}{\Gamma(\alpha n+1)}. \tag{61}$$

In general, the normalisation function $B(\alpha)$ can be any function satisfying $B(0)=B(1)=1$, but for the present work we shall assume that all values of $B(\alpha)$ are real and strictly positive. The reason for introducing this multiplier function, often taken simply to be identically 1, is because sometimes we may wish some values of α to contribute more than others, and hence we allow ourselves leeway to weight different values of α if needed.

To make the above definitions rigorous, we prove the following result which establishes domains of functions on which the ABR and ABC fractional differential operators are well-defined.

Lemma 2.4.3. *For given $\alpha,a,t\in\mathbb{R}$ with $a<t$ and $0<\alpha<1$, and a given normalisation function B, the ABR derivative ${}^{ABR}_{a}D^{\alpha}_{t}f(t)$ is well-defined for any function f such that the RL integral ${}_{a}D^{-\alpha}_{t}f(t)$ is well-defined, while the ABC derivative ${}^{ABC}_{a}D^{\alpha}_{t}f(t)$ is well-defined for any differentiable function f such that f' is an L^{1} function.*

Proof. We first consider the ABR derivative, and differentiate explicitly:

$$\begin{aligned}
{}^{ABR}_{a}D^{\alpha}_{t}f(t)&=\tfrac{B(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_{a}^{t}f(x)E_{\alpha}\left(\tfrac{-\alpha}{1-\alpha}(t-x)^{\alpha}\right)\mathrm{d}x\\
&=\frac{B(\alpha)}{1-\alpha}\left[f(t)E_{\alpha}\left(\tfrac{-\alpha}{1-\alpha}(0)^{\alpha}\right)+\int_{a}^{t}f(x)\frac{\mathrm{d}}{\mathrm{d}t}\left(E_{\alpha}\left(\tfrac{-\alpha}{1-\alpha}(t-x)^{\alpha}\right)\right)\mathrm{d}x\right]\\
&=\frac{B(\alpha)}{1-\alpha}\left[f(t)+\int_{a}^{t}f(x)\sum_{n=0}^{\infty}\frac{n\alpha\left(\tfrac{-\alpha}{1-\alpha}\right)^{n}(t-x)^{n\alpha-1}}{\Gamma(n\alpha+1)}\mathrm{d}x\right].
\end{aligned}\tag{62}$$

The function $E_\alpha\big(\frac{-\alpha}{1-\alpha}(t-x)^\alpha\big)$ and its t-derivative, considered as functions of x, are holomorphic at every point in the interval $[a,t)$. And the interval of integration is finite, so the only way the integral could possibly diverge would be due to behaviour near $x=t$. Thus the conditions for the ABR derivative to be well-defined are exactly that the integral in (62) should behave well as $x\to t$ from below.

As $x\to t$, we have $(t-x)^\alpha\to 0$ and therefore

$$E_\alpha\left(\tfrac{-\alpha}{1-\alpha}(t-x)^\alpha\right)\sim 1+\tfrac{-\alpha}{(1-\alpha)\Gamma(\alpha+1)}(t-x)^\alpha,$$

giving

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E_\alpha\left(\tfrac{-\alpha}{1-\alpha}(t-x)^\alpha\right)\right)\sim\tfrac{-\alpha^2}{(1-\alpha)\Gamma(\alpha+1)}(t-x)^{\alpha-1}.$$

So the integral in (62) converges if and only if

$$\int_a^t f(x)(t-x)^{\alpha-1}\,\mathrm{d}x$$

converges, i.e. if and only if the RL integral ${}_aD_t^{-\alpha}f(t)$ is well-defined.

Now we consider the ABC derivative:

$$ {}^{ABC}_{\ \ a}D_t^\alpha f(t)=\tfrac{B(\alpha)}{1-\alpha}\int_a^t f'(x)E_\alpha\left(\tfrac{-\alpha}{1-\alpha}(t-x)^\alpha\right)\mathrm{d}x. \tag{63}$$

Once again, the function $E_\alpha\big(\frac{-\alpha}{1-\alpha}(t-x)^\alpha\big)$ is holomorphic as a function of x at every point in the interval $[a,t)$, and the interval of integration is finite. So the conditions for the ABC derivative to be well-defined are exactly that this integral should behave well as $x\to t$ from below. We also know that as $x\to t$,

$$E_\alpha\left(\tfrac{-\alpha}{1-\alpha}(t-x)^\alpha\right)\sim 1,$$

so the integral in (63) converges if and only if

$$\int_a^t f'(x)\,\mathrm{d}x$$

converges, for which it suffices that f is differentiable and f' is L^1. $\square$

The result of Lemma 2.4.3 shows how the conditions on f in Definitions 2.4.1 and 2.4.2 were chosen: it is on these function spaces for f that the ABR and ABC fractional derivatives are well-defined. (It is shown in [127] that the RL integral is well-defined for $f\in L^1[a,b]$.)

Theorem 2.4.4. *The ABR fractional derivative can be expressed as*

$$ {}^{ABR}_{\ a}D^{\alpha}_{t} f(t) = \frac{B(\alpha)}{1-\alpha} \sum_{n=0}^{\infty} \left(\frac{-\alpha}{1-\alpha} \right)^{n} \frac{\mathrm{d}}{\mathrm{d}t} \left({}^{RL}_{\ a}I^{\alpha n+1}_{t} f(t) \right), \tag{64} $$

or equivalently (provided composition of RL differintegrals of f works as it should) as

$$ {}^{ABR}_{\ a}D^{\alpha}_{t} f(t) = \frac{B(\alpha)}{1-\alpha} \sum_{n=0}^{\infty} \left(\frac{-\alpha}{1-\alpha} \right)^{n} {}^{RL}_{\ a}I^{\alpha n}_{t} f(t), \tag{65} $$

each series converging locally uniformly in t for any a, α, f satisfying the conditions laid out in Definition 2.4.1.

Proof. The Mittag-Leffler function $E_{\alpha}(x)$ is an entire function of x, the series (61) converging locally uniformly in the whole complex plane. So the ABR fractional derivative can be rewritten as follows:

$$ \begin{aligned} {}^{ABR}_{\ a}D^{\alpha}_{t} f(t) &= \frac{B(\alpha)}{1-\alpha} \frac{\mathrm{d}}{\mathrm{d}t} \int_{a}^{t} f(x) \sum_{n=0}^{\infty} \frac{(-\alpha)^{n}(t-x)^{\alpha n}}{(1-\alpha)^{n}\Gamma(\alpha n+1)} \,\mathrm{d}x \\ &= \frac{B(\alpha)}{1-\alpha} \sum_{n=0}^{\infty} \left(\frac{-\alpha}{1-\alpha} \right)^{n} \frac{1}{\Gamma(\alpha n+1)} \frac{\mathrm{d}}{\mathrm{d}t} \int_{a}^{t} (t-x)^{\alpha n} f(x) \,\mathrm{d}x \\ &= \frac{B(\alpha)}{1-\alpha} \sum_{n=0}^{\infty} \left(\frac{-\alpha}{1-\alpha} \right)^{n} \frac{\mathrm{d}}{\mathrm{d}t} \left({}^{RL}_{\ a}I^{\alpha n+1}_{t} f(t) \right), \end{aligned} $$

where ${}^{RL}I$ is the standard Riemann–Liouville fractional integral. □

In fractional calculus, dealing with convergent series is a common requirement. The Grünwald–Letnikov differintegral in Definition 1.1.13 is expressed as a series, and so are the expressions of the fractional Leibniz rule and fractional chain rule in Lemmas 1.1.8 and 1.1.9. In this case, we have expressed the AB fractional derivatives as convergent series of RL fractional integrals. This means that in some sense, problems concerning the AB model can be reduced to problems in the classical RL model.

The new series formula may also be useful from a numerical point of view. The original formula for AB derivatives is written in terms of the transcendental Mittag-Leffler function, and any explicit calculations of AB derivatives would necessarily involve dealing with this function in some way. But with the new formula, we can obtain an approximation to the AB derivative by truncating the series after some finite number of terms and then using standard Riemann–Liouville numerical methods to estimate each term of the sum.

Theorem 2.4.4 enables us to derive the formula for Laplace transforms of ABR derivatives in a different way from [18]. The following identity is equation (9) in [18]:

$$\widehat{{}^{ABR}_{\ 0}D^{\alpha}_{t}}f(s)=\frac{B(\alpha)}{1-\alpha}\left(\frac{s^{\alpha}}{s^{\alpha}+\frac{\alpha}{1-\alpha}}\right)\hat{f}(s), \tag{66}$$

valid for any sufficiently well-behaved function f. We can now prove this by taking Laplace transforms directly on the series in (64):

$$\begin{aligned}
\widehat{{}^{ABR}_{\ 0}D^{\alpha}_{t}}f(s)&=\frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n}\left(\widehat{\frac{\mathrm{d}}{\mathrm{d}t}\,{}^{RL}_{\ 0}I^{\alpha n+1}_{t}f}\right)(s)\\
&=\frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n}\left(s\big(s^{-\alpha n-1}\hat{f}(s)\big)-{}^{RL}_{\ 0}I^{\alpha n+1}_{t}f(0)\right)\\
&=\frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\left[\left(\tfrac{-\alpha}{1-\alpha}s^{-\alpha}\right)^{n}\right]\hat{f}(s)-\frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n}{}^{RL}_{\ 0}I^{\alpha n+1}_{t}f(0)\\
&=\frac{B(\alpha)}{1-\alpha}\left(1-\tfrac{-\alpha}{1-\alpha}s^{-\alpha}\right)^{-1}\hat{f}(s)-{}^{ABR}_{\ 0}D^{\alpha}_{t}\circ{}^{RL}_{\ 0}I_{t}f(0)\\
&=\frac{B(\alpha)}{1-\alpha}\left(\frac{s^{\alpha}}{s^{\alpha}+\frac{\alpha}{1-\alpha}}\right)\hat{f}(s)-{}^{ABR}_{\ 0}D^{\alpha}_{t}\circ{}^{RL}_{\ 0}I_{t}f(0).
\end{aligned}$$

Thus we have derived the following more general formula for Laplace transforms of ABR derivatives, which reduces to (66) when f has sufficiently nice convergence properties at the lower limit $t=0$:

$$\widehat{{}^{ABR}_{\ 0}D^{\alpha}_{t}}f(s)=\frac{B(\alpha)}{1-\alpha}\left(\frac{s^{\alpha}}{s^{\alpha}+\frac{\alpha}{1-\alpha}}\right)\hat{f}(s)-{}^{ABR}_{\ 0}D^{\alpha}_{t}\circ{}^{RL}_{\ 0}I_{t}f(0). \tag{67}$$

Theorem 2.4.4 also has a number of other useful corollaries.

Corollary 2.4.5. *The AB fractional integral operator* ${}^{AB}_{\ a}I^{\alpha}$, *defined by* (58), *is both a left and right inverse to the ABR fractional differential operator* ${}^{ABR}_{\ a}D^{\alpha}$ *whenever* a,α,f *satisfy the conditions from Definition 2.4.1.*

Proof. First let us note that the expression (58) is well-defined if and only if the RL integral ${}^{RL}_{\ a}I^{\alpha}_{t}f(t)$ is well-defined, which matches with our assumptions on f for the αth ABR fractional derivative to be well-defined.

The expression (65) can be *formally* rewritten as

$${}^{ABR}_{\ a}D^{\alpha}_{t}f(t)=\frac{B(\alpha)}{1-\alpha}\left(1-\left(\tfrac{-\alpha}{1-\alpha}\right){}^{RL}_{\ a}I^{\alpha}_{t}\right)^{-1}f(t)=\left(\frac{1-\alpha}{B(\alpha)}+\frac{\alpha}{B(\alpha)}\,{}^{RL}_{\ a}I^{\alpha}_{t}\right)^{-1}f(t),$$

which gives us the motivation for using (58) as our definition for AB fractional integrals. Now we need to prove rigorously that

$${}^{ABR}_{\ a}D^{\alpha}_{t}\left({}^{AB}_{\ a}I^{\alpha}_{t}f(t)\right)=f(t) \tag{68}$$

and

$$ {}^{AB}_{\ a}I^{\alpha}_{t}\Big({}^{ABR}_{\quad a}D^{\alpha}_{t} f(t)\Big) = f(t) \tag{69}$$

for a, α, f as stated. We proceed as follows.

$$\begin{aligned}
{}^{ABR}_{\quad a}D^{\alpha}_{t}\Big({}^{AB}_{\ a}I^{\alpha}_{t} f(t)\Big) &= {}^{ABR}_{\quad a}D^{\alpha}_{t}\Big(\tfrac{1-\alpha}{B(\alpha)} f(t) + \tfrac{\alpha}{B(\alpha)}\, {}^{RL}_{\ a}I^{\alpha}_{t} f(t)\Big) \\
&= \frac{1-\alpha}{B(\alpha)}\, {}^{ABR}_{\quad a}D^{\alpha}_{t} f(t) + \frac{\alpha}{B(\alpha)}\, {}^{ABR}_{\quad a}D^{\alpha}_{t}\Big({}^{RL}_{\ a}I^{\alpha}_{t} f(t)\Big) \\
&= \sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n} {}^{RL}_{\ a}I^{\alpha n}_{t} f(t) \\
&\qquad\qquad + \frac{\alpha}{1-\alpha}\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n} {}^{RL}_{\ a}I^{\alpha n}_{t}\Big({}^{RL}_{\ a}I^{\alpha}_{t} f(t)\Big) \\
&= \sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n} {}^{RL}_{\ a}I^{\alpha n}_{t} f(t) - \sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n+1} {}^{RL}_{\ a}I^{\alpha n+\alpha}_{t} f(t) \\
&= f(t); \\
{}^{AB}_{\ a}I^{\alpha}_{t}\Big({}^{ABR}_{\quad a}D^{\alpha}_{t} f(t)\Big) &= \frac{1-\alpha}{B(\alpha)}\, {}^{ABR}_{\quad a}D^{\alpha}_{t} f(t) + \frac{\alpha}{B(\alpha)}\, {}^{RL}_{\ a}I^{\alpha}_{t}\Big({}^{ABR}_{\quad a}D^{\alpha}_{t} f(t)\Big) \\
&= \sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n} {}^{RL}_{\ a}I^{\alpha n}_{t} f(t) \\
&\qquad\qquad + \frac{\alpha}{1-\alpha}\, {}^{RL}_{\ a}I^{\alpha}_{t}\left(\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n} {}^{RL}_{\ a}I^{\alpha n}_{t} f(t)\right) \\
&= \sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n} {}^{RL}_{\ a}I^{\alpha n}_{t} f(t) - \sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n+1} {}^{RL}_{\ a}I^{\alpha+\alpha n}_{t} f(t) \\
&= f(t),
\end{aligned}$$

where in both cases we have used Lemma 1.1.6 on the composition properties of Riemann–Liouville fractional integrals. Note that since $f \in L^1(a,b)$, all RL fractional integrals of f are well-defined, and therefore so are the ABR derivative of f (by Lemma 2.4.3), the AB integral of f (by the definition (58)), and their compositions. □

Example 2.4.6. As an example to verify the result of Corollary 2.4.5, we set $f(t) = 1$ and calculate ${}^{AB}_{\ a}I^{\alpha}_{t}\Big({}^{ABR}_{\quad a}D^{\alpha}_{t} f(t)\Big)$ explicitly using the definitions (59) and (58). Firstly, by (59) we have

$$\begin{aligned}
{}^{ABR}_{a}D^{\alpha}_{t}(1) &= \frac{B(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t E_\alpha\left(\tfrac{-\alpha}{1-\alpha}(t-x)^\alpha\right)\mathrm{d}x \\
&= \frac{B(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=0}^{\infty}\frac{\left(\frac{-\alpha}{1-\alpha}\right)^n}{\Gamma(n\alpha+1)}\int_a^t (t-x)^{n\alpha}\,\mathrm{d}x \\
&= \frac{B(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\left((t-a)\sum_{n=0}^{\infty}\frac{\left[\frac{-\alpha}{1-\alpha}(t-a)^\alpha\right]^n}{\Gamma(n\alpha+2)}\right) \\
&= \tfrac{B(\alpha)}{1-\alpha}E_\alpha\left(\tfrac{-\alpha}{1-\alpha}(t-a)^\alpha\right).
\end{aligned}$$

Then by applying (58) to this, we have

$$\begin{aligned}
{}^{AB}_{a}I^{\alpha}_{t}\left({}^{ABR}_{a}D^{\alpha}_{t}(1)\right) &= \frac{1-\alpha}{B(\alpha)}\left({}^{ABR}_{a}D^{\alpha}_{t}(1)\right) + \frac{\alpha}{B(\alpha)}\,{}^{RL}_{a}I^{\alpha}_{t}\left({}^{ABR}_{a}D^{\alpha}_{t}(1)\right) \\
&= E_\alpha\left(\tfrac{-\alpha}{1-\alpha}(t-a)^\alpha\right) \\
&\qquad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^t (t-x)^{\alpha-1}\tfrac{B(\alpha)}{1-\alpha}E_\alpha\left(\tfrac{-\alpha}{1-\alpha}(x-a)^\alpha\right)\mathrm{d}x \\
&= E_\alpha\left(\tfrac{-\alpha}{1-\alpha}(t-a)^\alpha\right) \\
&\qquad + \frac{\alpha}{(1-\alpha)\Gamma(\alpha)}\sum_{n=0}^{\infty}\frac{\left(\frac{-\alpha}{1-\alpha}\right)^n}{\Gamma(n\alpha+1)}(t-a)^{(n+1)\alpha}B(\alpha,n\alpha+1) \\
&= E_\alpha\left(\tfrac{-\alpha}{1-\alpha}(t-a)^\alpha\right) - \sum_{n=0}^{\infty}\frac{\left(\frac{-\alpha}{1-\alpha}\right)^{n+1}(t-a)^{(n+1)\alpha}}{\Gamma((n+1)\alpha+1)} \\
&= E_\alpha\left(\tfrac{-\alpha}{1-\alpha}(t-a)^\alpha\right) - \left[E_\alpha\left(\tfrac{-\alpha}{1-\alpha}(t-a)^\alpha\right)-1\right] \\
&= 1,
\end{aligned}$$

precisely as expected. □

Note that the equation (69) should no longer be valid when $\alpha = 1$, because in this case the result for ordinary derivatives would be a Newton–Leibniz rule rather than a direct inverse relation. However, we can see from examining the above example that the αth ABR derivative does not always converge to the standard 1st derivative as $\alpha \to 1$. Specifically,

$$\begin{aligned}
\lim_{\alpha\to1}\left({}^{ABR}_{a}D^{\alpha}_{t}(1)\right) &= \lim_{\alpha\to1}\left(\tfrac{B(\alpha)}{1-\alpha}E_\alpha\left(\tfrac{-\alpha}{1-\alpha}(t-a)^\alpha\right)\right) \\
&= \lim_{\alpha\to1}\left(\tfrac{1}{1-\alpha}\exp\left(\tfrac{-(t-a)}{1-\alpha}\right)\right) = \delta(t-a),
\end{aligned}$$

the Dirac delta function. This is equal to zero (the 1st derivative of $f(t)$ in this case) almost everywhere, but the blowup at the limiting point $t = a$ changes the behaviour of the integral. In fact, we can observe the same behaviour with the classical Riemann–

Liouville derivative: in that model, the αth derivative of $f(t)=1$ is $\frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)}$, which again blows up at the limiting point $t=a$. In both models, the double limit as $\alpha\to 1$ and $t\to a$ needs to be handled with care. For the AB model, as $\alpha\to 1$, the ABR derivative converges to the standard derivative only *almost* everywhere (everywhere except $t=a$), and thus the Newton–Leibniz formula at $\alpha=1$ cannot be derived as a limit of the corresponding result (69) for $\alpha<1$.

In this way, we can understand the apparent discrepancy between the direct inverse relationship of the ABR derivative and AB integral and the Newton–Leibniz relationship of the standard 1st-order derivative and integral. See, however, Corollary 2.4.9 below for a valid Newton–Leibniz relationship between the ABC derivative and the AB integral.

Corollary 2.4.7. *The AB integral operators and ABR differential operators form a commutative family of differintegral operators:*

$$\begin{aligned} {}^{ABR}_{a}D^{\alpha}_{t}\Big({}^{ABR}_{a}D^{\beta}_{t}f(t)\Big) &= {}^{ABR}_{a}D^{\beta}_{t}\Big({}^{ABR}_{a}D^{\alpha}_{t}f(t)\Big), \\ {}^{AB}_{a}I^{\alpha}_{t}\Big({}^{AB}_{a}I^{\beta}_{t}f(t)\Big) &= {}^{AB}_{a}I^{\beta}_{t}\Big({}^{AB}_{a}I^{\alpha}_{t}f(t)\Big), \\ {}^{ABR}_{a}D^{\alpha}_{t}\Big({}^{AB}_{a}I^{\beta}_{t}f(t)\Big) &= {}^{AB}_{a}I^{\beta}_{t}\Big({}^{ABR}_{a}D^{\alpha}_{t}f(t)\Big), \end{aligned}$$

for $\alpha,\beta\in(0,1)$ and a,f satisfying the conditions from Definition 2.4.1.

Proof. For the first identity, we use equation (65) to get

$$\begin{aligned} {}^{ABR}_{a}D^{\alpha}_{t}\Big({}^{ABR}_{a}D^{\beta}_{t}f(t)\Big) &= \frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\Big(\frac{-\alpha}{1-\alpha}\Big)^{n}\,{}^{RL}_{a}I^{\alpha n}_{t}\left[\frac{B(\beta)}{1-\beta}\sum_{m=0}^{\infty}\Big(\frac{-\beta}{1-\beta}\Big)^{m}\,{}^{RL}_{a}I^{\beta m}_{t}f(t)\right] \\ &= \frac{B(\alpha)B(\beta)}{(1-\alpha)(1-\beta)}\sum_{m,n}\frac{(-\alpha)^{n}(-\beta)^{m}}{(1-\alpha)^{n}(1-\beta)^{m}}\,{}^{RL}_{a}I^{\alpha n+\beta m}_{t}f(t). \end{aligned}$$

This expression is symmetric in α and β – it remains the same if these two variables are swapped – so it must be equal to ${}^{ABR}_{a}D^{\beta}_{t}\big({}^{ABR}_{a}D^{\alpha}_{t}f(t)\big)$, as required.

For the second identity, we use equation (58) to get

$${}^{AB}_{a}I^{\alpha}_{t}\Big({}^{AB}_{a}I^{\beta}_{t}f(t)\Big) = \left(\frac{1-\alpha}{B(\alpha)}+\frac{\alpha}{B(\alpha)}\,{}^{RL}_{a}I^{\alpha}_{t}\right)\left(\frac{1-\beta}{B(\beta)}+\frac{\beta}{B(\beta)}\,{}^{RL}_{a}I^{\beta}_{t}\right)f(t),$$

which again is symmetric in α and β, since the RL integral operators commute.

For the third identity, we use equations (65) and (58) together:

$$
\begin{aligned}
{}^{ABR}_{\ a}D^{\alpha}_{t}\left({}^{AB}_{\ a}I^{\beta}_{t}f(t)\right) &= \frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n}\,{}^{RL}_{\ a}I^{\alpha n}_{t}\left[\left(\frac{1-\beta}{B(\beta)}+\frac{\beta}{B(\beta)}\,{}^{RL}_{\ a}I^{\beta}_{t}\right)f(t)\right] \\
&= \left(\frac{1-\beta}{B(\beta)}+\frac{\beta}{B(\beta)}\,{}^{RL}_{\ a}I^{\beta}_{t}\right)\left[\frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n}\,{}^{RL}_{\ a}I^{\alpha n}_{t}f(t)\right] \\
&= {}^{AB}_{\ a}I^{\beta}_{t}\left({}^{ABR}_{\ a}D^{\alpha}_{t}f(t)\right),
\end{aligned}
$$

again using the fact that RL integral operators are commutative, as well as the local uniform convergence of the series in (65). □

Note that both of the above corollaries could also be proved using the Laplace transform formula (66) which was established in [18]. But the advantage of the new approach, proving them directly from Theorem 2.4.4, is that it works for *all* functions f such that the AB fractional derivatives and integrals are well-defined, not just those f which have well-defined Laplace transforms. The proofs are more direct, without the need to pass back and forth between the time domain and the frequency domain.

We also have the following analogous result for ABC fractional derivatives.

Theorem 2.4.8. *The ABC fractional derivative can be expressed as*

$$
{}^{ABC}_{\ a}D^{\alpha}_{t}f(t) = \frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n}\,{}^{RL}_{\ a}I^{\alpha n+1}_{t}f'(t), \tag{70}
$$

this series converging locally uniformly in t for any a,α,f satisfying the conditions from Definition 2.4.2.

Proof. As for Theorem 2.4.4. □

And just as we did above for ABR derivatives, we can use this result to derive the following expression for Laplace transforms of ABC derivatives:

$$
\widehat{{}^{ABC}_{\ a}D^{\alpha}_{t}}f(s) = \frac{B(\alpha)}{1-\alpha}\cdot\frac{s^{\alpha-1}}{s^{\alpha}+\frac{\alpha}{1-\alpha}}\left(s\hat{f}(s)-f(0)\right) \tag{71}
$$

for any function f with well-defined Laplace transform $\hat{f}$ and ABC fractional derivative ${}^{ABC}_{\ a}D^{\alpha}_{t}f(t)$. This was equation (10) in [18].

The following result was already shown in [37] using Laplace transforms, but we can now prove it in a much more elementary way, with fewer required assumptions on the function f, by using the series formula from Theorem 2.4.8.

Corollary 2.4.9. *The AB fractional integral and the ABC fractional derivative satisfy the following Newton–Leibniz formula:*

$$ {}^{AB}_{\ a}I^{\alpha}_{t}\Big({}^{ABC}_{\ \ a}D^{\alpha}_{t} f(t)\Big) = f(t) - f(a), \tag{72}$$

valid whenever a, α, f satisfy the conditions from Definition 2.4.2.

Proof. By the definition (58) of the AB integral, together with the series formula (70) for the ABC derivative, we have

$$\begin{aligned}
{}^{AB}_{\ a}I^{\alpha}_{t}\Big({}^{ABC}_{\ \ a}D^{\alpha}_{t} f(t)\Big) &= \frac{1-\alpha}{B(\alpha)}\,{}^{ABC}_{\ \ a}D^{\alpha}_{t} f(t) + \frac{\alpha}{B(\alpha)}\,{}^{RL}_{\ a}I^{\alpha}_{t}\Big({}^{ABC}_{\ \ a}D^{\alpha}_{t} f(t)\Big) \\
&= \sum_{n=0}^{\infty} \left(\tfrac{-\alpha}{1-\alpha}\right)^{n} {}^{RL}_{\ a}I^{\alpha n+1}_{t} f'(t) \\
&\qquad\qquad + \frac{\alpha}{1-\alpha}\,{}^{RL}_{\ a}I^{\alpha}_{t}\left(\sum_{n=0}^{\infty} \left(\tfrac{-\alpha}{1-\alpha}\right)^{n} {}^{RL}_{\ a}I^{\alpha n+1}_{t} f'(t)\right) \\
&= \sum_{n=0}^{\infty} \left(\tfrac{-\alpha}{1-\alpha}\right)^{n} {}^{RL}_{\ a}I^{\alpha n+1}_{t} f'(t) - \sum_{n=0}^{\infty} \left(\tfrac{-\alpha}{1-\alpha}\right)^{n+1} {}^{RL}_{\ a}I^{\alpha+\alpha n+1}_{t} f'(t) \\
&= {}^{RL}_{\ a}I^{1}_{t} f'(t) = f(t) - f(a),
\end{aligned}$$

by the standard Newton–Leibniz formula. □

This is significant because the Newton–Leibniz formula is a required element in the derivation of a theory of fractional vector calculus [134]. Thus, knowing that a Newton–Leibniz analogue holds for ABC derivatives may enable us to construct a theory of fractional vector calculus in the AB model too.

2.4.3 Some linear ordinary differential equations

Here and in §2.4.4, we consider certain classes of ordinary differential equations (ODEs) in the Atangana–Baleanu model, and different methods which can be used to solve them.

As a basic first case, consider the following simple fractional ODE:

$$ {}^{ABR}_{\ \ 0}D^{\alpha}_{t} f(t) - \tfrac{B(\alpha)}{1-\alpha} f(t) = g(t) \tag{73}$$

where f and g are Laplace-transformable functions and $\alpha \in (0,1)$. Taking Laplace transforms of (73), using the formula (67), yields

$$ \tfrac{B(\alpha)}{1-\alpha}\left(\frac{s^{\alpha}}{s^{\alpha}+\frac{\alpha}{1-\alpha}}\right)\hat{f}(s) - {}^{ABR}_{\ \ 0}D^{\alpha}_{t} \circ {}^{RL}_{\ 0}I_{t} f(0) - \tfrac{B(\alpha)}{1-\alpha}\hat{f}(s) = \hat{g}(s), $$

and therefore

$$\tfrac{B(\alpha)}{1-\alpha}\left(\frac{\frac{-\alpha}{1-\alpha}}{s^\alpha+\frac{\alpha}{1-\alpha}}\right)\hat{f}(s)=\hat{g}(s)+{}^{ABR}_{0}D^\alpha_t\circ{}^{RL}_{0}I_tf(0).$$

Thus the Laplace-transformed solution $\hat{f}(s)$ is given by

$$\hat{f}(s)=\tfrac{-(1-\alpha)^2}{\alpha B(\alpha)}\left(s^\alpha+\tfrac{\alpha}{1-\alpha}\right)\hat{h}(s),$$

where the function $\hat{h}(s)$ is defined by

$$\hat{h}(s)=\hat{g}(s)+{}^{ABR}_{0}D^\alpha_t\circ{}^{RL}_{0}I_tf(0)$$

or equivalently

$$h(t)=g(t)+\left[{}^{ABR}_{0}D^\alpha_t\circ{}^{RL}_{0}I_tf(0)\right]\delta(t). \tag{74}$$

Now the Laplace transform of the Riemann–Liouville fractional derivative ${}^{RL}_{0}D^\alpha_th(t)$ is $s^\alpha\hat{h}(s)-{}^{RL}_{0}I^{1-\alpha}_th(0)$, so the Laplace transform of ${}^{RL}_{0}D^\alpha_th(t)+{}^{RL}_{0}I^{1-\alpha}_th(0)\delta(t)$ is simply $s^\alpha\hat{h}(s)$. Thus the unique Laplace-transformable solution to equation (73) is

$$f(t)=\tfrac{-(1-\alpha)^2}{\alpha B(\alpha)}\left({}^{RL}_{0}D^\alpha_th(t)+{}^{RL}_{0}I^{1-\alpha}_th(0)\delta(t)\right)-\tfrac{1-\alpha}{B(\alpha)}h(t),$$

where h is defined by (74) and therefore depends only – and linearly – on g and initial values of f.

More generally, let us consider the following fractional ODE, inhomogeneous with arbitrary constant coefficients:

$${}^{ABR}_{0}D^\alpha_tf(t)-Af(t)=g(t) \tag{75}$$

where f and g are Laplace-transformable functions, $\alpha\in(0,1)$, and A is a constant. Write $k:=\frac{1-\alpha}{B(\alpha)}A$ for ease of notation, so that the equation (75) becomes

$${}^{ABR}_{0}D^\alpha_tf(t)-\tfrac{B(\alpha)}{1-\alpha}kf(t)=g(t).$$

Then take Laplace transforms of this equation, using the formula (67), to get

$$\frac{B(\alpha)}{1-\alpha}\left(\frac{s^\alpha-k[s^\alpha+\frac{\alpha}{1-\alpha}]}{s^\alpha+\frac{\alpha}{1-\alpha}}\right)\hat{f}(s)-{}^{ABR}_{0}D^\alpha_t\circ{}^{RL}_{0}I_tf(0)=\hat{g}(s).$$

Thus the Laplace-transformed solution $\hat{f}(s)$ can be written as:

$$\begin{aligned}
\hat{f}(s) &= \frac{1-\alpha}{B(\alpha)}\Big(s^{\alpha}+\tfrac{\alpha}{1-\alpha}\Big)\Big((1-k)s^{\alpha}-\tfrac{k\alpha}{1-\alpha}\Big)^{-1}\Big(\hat{g}(s)+{}^{ABR}_{\ \ 0}D^{\alpha}_{t}\circ{}^{RL}_{\ \ 0}I_{t}f(0)\Big) \\
&= \frac{1-\alpha}{(1-k)B(\alpha)}\Big(1+\tfrac{\alpha}{1-\alpha}s^{-\alpha}\Big)\Big(1-\tfrac{k\alpha}{(1-k)(1-\alpha)}s^{-\alpha}\Big)^{-1}\hat{h}(s) \\
&= \frac{1-\alpha}{(1-k)B(\alpha)}\Big(1+\tfrac{k\alpha}{(1-k)(1-\alpha)}s^{-\alpha}+\tfrac{k^2\alpha^2}{(1-k)^2(1-\alpha)^2}s^{-2\alpha}+\dots\Big)\Big(1+\tfrac{\alpha}{1-\alpha}s^{-\alpha}\Big)\hat{h}(s),
\end{aligned}$$

where h is defined by (74) as before.

From here, a simple way of proceeding yields

$$\hat{f}(s)=\tfrac{1}{k}\Big(1+\tfrac{k\alpha}{(1-k)(1-\alpha)}s^{-\alpha}+\tfrac{k^2\alpha^2}{(1-k)^2(1-\alpha)^2}s^{-2\alpha}+\dots\Big)\widehat{{}^{AB}_{\ 0}I^{\alpha}_{t}h}(s)$$

and therefore

$$\begin{aligned}
f(t) &= \frac{1}{1-k}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t E_{\alpha}\Big(\tfrac{k\alpha}{(1-k)(1-\alpha)}(t-x)^{\alpha}\Big)\,{}^{AB}_{\ 0}I^{\alpha}_{x}h(x)\,\mathrm{d}x \\
&= \frac{1}{1-k}\,{}^{AB}_{\ 0}I^{\alpha}_{t}h(t)+\frac{1}{1-k}\int_0^t\frac{\mathrm{d}}{\mathrm{d}t}E_{\alpha}\Big(\tfrac{k\alpha}{(1-k)(1-\alpha)}(t-x)^{\alpha}\Big)\,{}^{AB}_{\ 0}I^{\alpha}_{x}h(x)\,\mathrm{d}x. \qquad (76)
\end{aligned}$$

Alternatively, a slightly more interesting way of proceeding yields

$$\begin{aligned}
\hat{f}(s) &= \frac{1-\alpha}{(1-k)B(\alpha)}\Bigg(1+\Big[\tfrac{\alpha}{1-\alpha}+\tfrac{k\alpha}{(1-k)(1-\alpha)}\Big]s^{-\alpha} \\
&\qquad\qquad +\Big[\tfrac{k\alpha^2}{(1-k)(1-\alpha)^2}+\tfrac{k^2\alpha^2}{(1-k)^2(1-\alpha)^2}\Big]s^{-2\alpha}+\dots\Bigg)\hat{h}(s) \\
&= \frac{1-\alpha}{(1-k)B(\alpha)}\Big(1+\tfrac{\alpha}{(1-k)(1-\alpha)}s^{-\alpha}+\tfrac{k\alpha^2}{(1-k)^2(1-\alpha)^2}s^{-2\alpha}+\dots\Big)\hat{h}(s) \\
&= \frac{1-\alpha}{k(1-k)B(\alpha)}\Big((k-1)+1+\tfrac{k\alpha}{(1-k)(1-\alpha)}s^{-\alpha}+\tfrac{k^2\alpha^2}{(1-k)^2(1-\alpha)^2}s^{-2\alpha}+\dots\Big)\hat{h}(s)
\end{aligned}$$

and therefore

$$\begin{aligned}
f(t) &= \frac{1-\alpha}{-kB(\alpha)}h(t)+\frac{1-\alpha}{k(1-k)B(\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t E_{\alpha}\Big(\tfrac{k\alpha}{(1-k)(1-\alpha)}(t-x)^{\alpha}\Big)h(x)\,\mathrm{d}x \\
&= -\frac{1}{A}h(t)+\frac{1}{A(1-k)}\left[h(t)+\int_0^t\frac{\mathrm{d}}{\mathrm{d}t}E_{\alpha}\Big(\tfrac{k\alpha}{(1-k)(1-\alpha)}(t-x)^{\alpha}\Big)h(x)\,\mathrm{d}x\right] \\
&= \frac{k}{A(1-k)}h(t)+\frac{1}{A(1-k)}\int_0^t\frac{\mathrm{d}}{\mathrm{d}t}E_{\alpha}\Big(\tfrac{k\alpha}{(1-k)(1-\alpha)}(t-x)^{\alpha}\Big)h(x)\,\mathrm{d}x. \qquad (77)
\end{aligned}$$

So the (equivalent) expressions (76) and (77), where the function h is defined by (74), provide the unique Laplace-transformable solution to the ODE (75).

By applying this argument multiple times, we can solve any linear sequential fractional

ODE of the form

$$\left({}^{ABR}_{0}D^{\alpha}_{t}-A\right)\left({}^{ABR}_{0}D^{\beta}_{t}-B\right)\dots\left({}^{ABR}_{0}D^{\gamma}_{t}-C\right)f(t)=g(t), \tag{78}$$

where g is a Laplace-transformable function and $\alpha,\beta,\dots,\gamma\in(0,1)$ and $A,B,\dots,C$ are constants. In each case, we can construct a nested integral formula for the unique Laplace-transformable solution f to the ODE (78).

Example 2.4.10. As an example application of this method, let us consider the following sequential fractional ODE:

$${}^{ABR}_{0}D^{1/2}_{t}\circ{}^{ABR}_{0}D^{1/2}_{t}f(t)=f(t)+g(t), \tag{79}$$

where we take the normalisation function $B(\alpha)$ to be identically 1. This ODE can be rewritten as

$$\left({}^{ABR}_{0}D^{1/2}_{t}-1\right)\left({}^{ABR}_{0}D^{1/2}_{t}+1\right)f(t)=g(t),$$

which we can split into a coupled pair of ODEs as follows:

$${}^{ABR}_{0}D^{1/2}_{t}j(t)-j(t)=g(t); \tag{80}$$

$${}^{ABR}_{0}D^{1/2}_{t}f(t)+f(t)=j(t). \tag{81}$$

Note that both (80) and (81) are ODEs in the form of (75). In the first case, we have $\alpha=1/2$, $A=1$, and therefore $k=1/2$, so the formula (77) becomes

$$j(t)=g(t)+2\int_0^t\frac{\mathrm{d}}{\mathrm{d}t}E_{1/2}\big((t-x)^{1/2}\big)g(x)\,\mathrm{d}x.$$

In the second case, we have $\alpha=1/2$, $A=-1$, and therefore $k=-1/2$, so the formula (77) becomes

$$f(t)=\frac{1}{3}j(t)-\frac{2}{3}\int_0^t\frac{\mathrm{d}}{\mathrm{d}t}E_{1/2}\Big(-\tfrac{1}{3}(t-x)^{1/2}\Big)j(x)\,\mathrm{d}x.$$

Thus the solution to the sequential ODE (79) is:

$$\begin{aligned}f(t)=\frac{1}{3}g(t)+\frac{2}{3}\int_0^t\frac{\mathrm{d}}{\mathrm{d}t}E_{1/2}\big((t-x)^{1/2}\big)g(x)\,\mathrm{d}x-\frac{2}{3}\int_0^t\frac{\mathrm{d}}{\mathrm{d}t}E_{1/2}\Big(-\tfrac{1}{3}(t-x)^{1/2}\Big)g(x)\,\mathrm{d}x\\-\frac{2}{3}\int_0^t\frac{\mathrm{d}}{\mathrm{d}t}E_{1/2}\Big(-\tfrac{1}{3}(t-x)^{1/2}\Big)\left[2\int_0^x\frac{\mathrm{d}}{\mathrm{d}x}E_{1/2}\big((x-y)^{1/2}\big)g(y)\,\mathrm{d}y\right]\mathrm{d}x.\end{aligned}$$

□

Now let us consider the same types of ODEs but with derivatives of Caputo type instead of Riemann–Liouville type. The solution in this case runs in much the same way as before, except that now the initial values are slightly easier to deal with.

As a basic first case, let us consider the following simple fractional ODE, analogous to (73):

$$ {}^{ABC}_{0}D^{\alpha}_{t}f(t)-\tfrac{B(\alpha)}{1-\alpha}f(t)=g(t) \tag{82}$$

where f and g are Laplace-transformable functions and $\alpha\in(0,1)$. Taking Laplace transforms, we get

$$\begin{aligned}
(82)\Leftrightarrow\ &\frac{B(\alpha)}{1-\alpha}\left(\frac{s^{\alpha-1}}{s^{\alpha}+\frac{\alpha}{1-\alpha}}\right)\Big(s\hat{f}(s)-f(0)\Big)-\frac{B(\alpha)}{1-\alpha}\hat{f}(s)=\hat{g}(s)\\
\Leftrightarrow\ &\frac{B(\alpha)}{1-\alpha}\cdot\frac{1}{s^{\alpha}+\frac{\alpha}{1-\alpha}}\Big(s^{\alpha}\hat{f}(s)-s^{\alpha-1}f(0)-\big(s^{\alpha}+\tfrac{\alpha}{1-\alpha}\big)\hat{f}(s)\Big)=\hat{g}(s)\\
\Leftrightarrow\ &\tfrac{B(\alpha)}{1-\alpha}\Big(\tfrac{-\alpha}{1-\alpha}\hat{f}(s)-s^{\alpha-1}f(0)\Big)=\big(s^{\alpha}+\tfrac{\alpha}{1-\alpha}\big)\hat{g}(s)\\
\Leftrightarrow\ &\tfrac{-\alpha B(\alpha)}{(1-\alpha)^2}\hat{f}(s)=\big(s^{\alpha}+\tfrac{\alpha}{1-\alpha}\big)\hat{g}(s)+\tfrac{B(\alpha)}{1-\alpha}s^{\alpha-1}f(0)\\
\Leftrightarrow\ &\hat{f}(s)=\tfrac{\alpha-1}{B(\alpha)}\hat{g}(s)-\tfrac{(1-\alpha)^2}{\alpha B(\alpha)}s^{\alpha}\hat{g}(s)+\tfrac{\alpha-1}{\alpha}s^{\alpha-1}f(0)\\
\Leftrightarrow\ &f(t)=\tfrac{\alpha-1}{B(\alpha)}g(t)-\tfrac{(1-\alpha)^2}{\alpha B(\alpha)}\Big({}^{RL}_{0}D^{\alpha}_{t}g(t)+{}^{RL}_{0}I^{1-\alpha}_{t}g(0)\delta(t)\Big)+\tfrac{\alpha-1}{\alpha}\cdot\tfrac{t^{-\alpha}}{\Gamma(1-\alpha)}f(0),
\end{aligned}$$

where ${}^{RL}D$ is the Riemann–Liouville fractional derivative. So a solution to the ODE (82), which is unique among Laplace-transformable functions, is

$$f(t)=\tfrac{\alpha-1}{B(\alpha)}g(t)-\tfrac{(1-\alpha)^2}{\alpha B(\alpha)}\,{}^{RL}_{0}D^{\alpha}_{t}g(t)-\Big(\tfrac{(1-\alpha)^2}{\alpha B(\alpha)}\,{}^{RL}_{0}I^{1-\alpha}_{t}g(0)\Big)\delta(t)+\tfrac{\alpha-1}{\alpha\Gamma(1-\alpha)}t^{-\alpha}f(0).$$

More generally, let us consider the following fractional ODE, inhomogeneous with arbitrary constant coefficients, identical to (75) except with ABC derivatives instead of ABR:

$$ {}^{ABC}_{0}D^{\alpha}_{t}f(t)-Af(t)=g(t) \tag{83}$$

where f and g are Laplace-transformable functions, $\alpha\in(0,1)$, and A is a constant. Note that this is the AB equivalent of a class of fractional ODEs which has been much studied in the classical Caputo case; see for example [63]. Write $k:=\frac{1-\alpha}{B(\alpha)}A$ again, so that equation (83) becomes

$$ {}^{ABC}_{0}D^{\alpha}_{t}f(t)-\tfrac{B(\alpha)}{1-\alpha}kf(t)=g(t).$$

Then take Laplace transforms of this, to get

$$\frac{B(\alpha)}{1-\alpha}\left(\frac{s^{\alpha}\hat{f}(s)-s^{\alpha-1}f(0)-k\big[s^{\alpha}+\frac{\alpha}{1-\alpha}\big]\hat{f}(s)}{s^{\alpha}+\frac{\alpha}{1-\alpha}}\right)=\hat{g}(s)$$

and therefore

$$\hat{f}(s)\Big((1-k)s^\alpha - \tfrac{k\alpha}{1-\alpha}\Big) - s^{\alpha-1}f(0) = \tfrac{1-\alpha}{B(\alpha)}\big(s^\alpha + \tfrac{\alpha}{1-\alpha}\big)\hat{g}(s).$$

Thus the Laplace-transformed solution $\hat{f}(s)$ is given by

$$\begin{aligned}
\hat{f}(s) &= \Big((1-k)s^\alpha - \tfrac{k\alpha}{1-\alpha}\Big)^{-1}\Big[\tfrac{1-\alpha}{B(\alpha)}\Big(s^\alpha + \tfrac{\alpha}{1-\alpha}\Big)\hat{g}(s) + s^{\alpha-1}f(0)\Big] \\
&= \Big(1 - \tfrac{k\alpha}{(1-k)(1-\alpha)}s^{-\alpha}\Big)^{-1}\Big[\tfrac{1-\alpha}{(1-k)B(\alpha)}\Big(1 + \tfrac{\alpha}{1-\alpha}s^{-\alpha}\Big)\hat{g}(s) + \tfrac{1}{(1-k)s}f(0)\Big] \\
&= \Big(1 + \tfrac{k\alpha}{(1-k)(1-\alpha)}s^{-\alpha} + \tfrac{k^2\alpha^2}{(1-k)^2(1-\alpha)^2}s^{-2\alpha} + \dots\Big)\times \\
&\qquad\qquad \Big[\tfrac{1-\alpha}{(1-k)B(\alpha)}\Big(1 + \tfrac{\alpha}{1-\alpha}s^{-\alpha}\Big)\hat{g}(s) + \tfrac{1}{(1-k)s}f(0)\Big].
\end{aligned}$$

Now we already know, from the work done above after equation (75), that the inverse Laplace transform of the $\hat{g}(s)$ part of the RHS is given by the (equivalent) expressions (76) and (77), except with h replaced by g in each case. And the $f(0)$ part comes to

$$\tfrac{1}{1-k}\sum_{n=0}^{\infty}\Big(\tfrac{k\alpha}{(1-k)(1-\alpha)}\Big)^n s^{-n\alpha-1}f(0),$$

so its inverse Laplace transform is

$$\tfrac{1}{1-k}\sum_{n=0}^{\infty}\Big(\tfrac{k\alpha}{(1-k)(1-\alpha)}\Big)^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}f(0) = \tfrac{1}{1-k}E_\alpha\Big(\tfrac{k\alpha}{(1-k)(1-\alpha)}t^\alpha\Big)f(0).$$

Thus, using equation (77) for the $\hat{g}(s)$ part, we find that the unique Laplace-transformable solution to the ODE (75) is given by

$$\begin{aligned}
f(t) &= -\frac{1}{A}g(t) + \frac{1}{A(1-k)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t E_\alpha\Big(\tfrac{k\alpha}{(1-k)(1-\alpha)}(t-x)^\alpha\Big)g(x)\,\mathrm{d}x \\
&\qquad\qquad + \frac{1}{1-k}E_\alpha\Big(\tfrac{k\alpha}{(1-k)(1-\alpha)}t^\alpha\Big)f(0) \\
&= \frac{k}{A(1-k)}g(t) + \frac{1}{A(1-k)}\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}E_\alpha\Big(\tfrac{k\alpha}{(1-k)(1-\alpha)}(t-x)^\alpha\Big)g(x)\,\mathrm{d}x \\
&\qquad\qquad + \frac{1}{1-k}E_\alpha\Big(\tfrac{k\alpha}{(1-k)(1-\alpha)}t^\alpha\Big)f(0).
\end{aligned} \tag{84}$$

By applying the above argument multiple times, we can solve any linear sequential fractional ODE of the form

$$\Big({}^{ABC}_{\ \ 0}D^\alpha_t - A\Big)\Big({}^{ABC}_{\ \ 0}D^\beta_t - B\Big)\dots\Big({}^{ABC}_{\ \ 0}D^\gamma_t - C\Big)f(t) = g(t), \tag{85}$$

where g is a Laplace-transformable function and $\alpha, \beta, \ldots, \gamma \in (0,1)$ and $A, B, \ldots, C$ are constants. In each case, we can construct a nested integral formula for the unique Laplace-transformable solution f to the ODE (85), which depends linearly on the initial condition $f(0)$.

Example 2.4.11. As an example application of this method, let us consider the following sequential fractional ODE, analogous to (79) but with ABC derivatives instead of ABR:

$$ {}^{ABC}_{0}D^{1/2}_t \circ {}^{ABC}_{0}D^{1/2}_t f(t) = f(t) + g(t), \tag{86} $$

where we take the normalisation function $B(\alpha)$ to be identically 1. As in Example 2.4.10, we can split this ODE into a coupled pair as follows:

$$ {}^{ABC}_{0}D^{1/2}_t j(t) - j(t) = g(t); \tag{87} $$

$$ {}^{ABC}_{0}D^{1/2}_t f(t) + f(t) = j(t). \tag{88} $$

Note that both (87) and (88) are ODEs in the form of (83). In the first case, we have $\alpha = 1/2$, $A = 1$, and therefore $k = 1/2$, so the formula (84) becomes

$$ j(t) = g(t) + 2\int_0^t \frac{\mathrm{d}}{\mathrm{d}t} E_{1/2}\big((t-x)^{1/2}\big) g(x)\,\mathrm{d}x + 2E_{1/2}\big(t^{1/2}\big) j(0). $$

In the second case, we have $\alpha = 1/2$, $A = -1$, and therefore $k = -1/2$, so the formula (84) becomes

$$ f(t) = \frac{1}{3} j(t) - \frac{2}{3}\int_0^t \frac{\mathrm{d}}{\mathrm{d}t} E_{1/2}\Big(-\tfrac{1}{3}(t-x)^{1/2}\Big) j(x)\,\mathrm{d}x + \frac{2}{3}E_{1/2}\Big(-\tfrac{1}{3}t^{1/2}\Big) f(0). $$

So with initial conditions giving $f(0) = j(0) = 0$, for example, the solution to the sequential ODE (79) is:

$$ \begin{aligned} f(t) = \frac{1}{3} g(t) &+ \frac{2}{3}\int_0^t \frac{\mathrm{d}}{\mathrm{d}t} E_{1/2}\big((t-x)^{1/2}\big) g(x)\,\mathrm{d}x - \frac{2}{3}\int_0^t \frac{\mathrm{d}}{\mathrm{d}t} E_{1/2}\Big(-\tfrac{1}{3}(t-x)^{1/2}\Big) g(x)\,\mathrm{d}x \\ &- \frac{2}{3}\int_0^t \frac{\mathrm{d}}{\mathrm{d}t} E_{1/2}\Big(-\tfrac{1}{3}(t-x)^{1/2}\Big) \left[2\int_0^x \frac{\mathrm{d}}{\mathrm{d}x} E_{1/2}\big((x-y)^{1/2}\big) g(y)\,\mathrm{d}y\right] \mathrm{d}x. \end{aligned} $$

Thus we see that the only difference between the results for linear ODEs in the ABR model and in the ABC model is in how the initial conditions manifest themselves in the solutions. □

Note that various examples of ODEs of the form (75) and (83) have already found applications in the real world, for example to electrical circuits [69]. Our analysis goes beyond these to cover general ODEs of the form (78) and (85).

2.4.4 Some nonlinear ordinary differential equations

Similar methods to those utilised above can also be used to solve certain special classes of nonlinear ordinary differential equations. For example, consider the following ODE, a fractional version of one of the classes of ODE considered in [6]:

$$ {}^{ABC}_{0}D^{\alpha}_{t}f(t)-Af*f(t)=g(t), \tag{89}$$

where f and g are Laplace-transformable functions, $\alpha\in(0,1)$, A is a constant, and $*$ denotes the convolution operation. Taking Laplace transforms of this equation, we get:

$$\frac{B(\alpha)}{1-\alpha}\cdot\frac{s^{\alpha-1}}{s^{\alpha}+\frac{\alpha}{1-\alpha}}\Big(s\hat{f}(s)-f(0)\Big)-A\big(\hat{f}(s)\big)^2=\hat{g}(s),$$

which can be rearranged to

$$A\big(\hat{f}(s)\big)^2-\frac{B(\alpha)}{1-\alpha}\cdot\frac{s^{\alpha}}{s^{\alpha}+\frac{\alpha}{1-\alpha}}\hat{f}(s)+\Big(\hat{g}(s)+\frac{B(\alpha)}{1-\alpha}\cdot\frac{s^{\alpha-1}}{s^{\alpha}+\frac{\alpha}{1-\alpha}}f(0)\Big)=0.$$

But this is simply a quadratic equation in $\hat{f}(s)$, so we can solve it to find:

$$\hat{f}(s)=\frac{\frac{B(\alpha)}{1-\alpha}\cdot\frac{s^{\alpha}}{s^{\alpha}+\frac{\alpha}{1-\alpha}}\pm\sqrt{\Big(\frac{B(\alpha)}{1-\alpha}\cdot\frac{s^{\alpha}}{s^{\alpha}+\frac{\alpha}{1-\alpha}}\Big)^2-4A\Big(\hat{g}(s)+\frac{B(\alpha)}{1-\alpha}\cdot\frac{s^{\alpha-1}}{s^{\alpha}+\frac{\alpha}{1-\alpha}}f(0)\Big)}}{2A}. \tag{90}$$

Note that the right hand side of (90) depends only on the function g and the constant $f(0)$, so we can take inverse Laplace transforms to get an explicit formula for $f(t)$.

Very similar arguments can of course be applied to the ODE (89) with the ABC derivative replaced by an ABR one, and the final result would look similar to (90) except with different dependence on the initial conditions.

As we can see, the final results are less succinct and elegant than those obtained above for linear ODEs, but this is natural since nonlinear ODEs almost always present extra difficulties as compared to the simpler linear case.

There are also some classes of nonlinear ODEs for which the series formula of Theorem 2.4.4 enables us to obtain a quick solution. For example, consider the following nonlinear ODE, a case of the Riccati equation, which in the classical Caputo context was considered in [42]:

$$ {}^{ABC}_{0}D^{\alpha}_{t}f(t)=P+Q\big(f(t)\big)^2,\quad f(0)=f_0 \tag{91}$$

We consider the possibility of a convergent series solution of the form

$$f(t)=\sum_{k=0}^{\infty}a_k t^{k\alpha}, \tag{92}$$

whose ABC derivative can be written as follows, using the result of Theorem 2.4.8.

$$\begin{aligned}
{}^{ABC}_{\ \ 0}D^{\alpha}_{t}f(t) &= \frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n}\,{}^{RL}_{\ 0}I^{n\alpha+1}_{t}\left(\sum_{k=1}^{\infty}a_k k\alpha t^{k\alpha-1}\right)\\
&= \frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n}\frac{a_k\Gamma(k\alpha+1)}{\Gamma((k+n)\alpha+1)}t^{(k+n)\alpha}\\
&= \sum_{m=1}^{\infty}\left[\frac{B(\alpha)}{1-\alpha}\sum_{k=1}^{m}a_k\left(\tfrac{-\alpha}{1-\alpha}\right)^{m-k}\frac{\Gamma(k\alpha+1)}{\Gamma(m\alpha+1)}\right]t^{m\alpha}.
\end{aligned}$$

Meanwhile, the right hand side of (91), with the ansatz (92), is simply

$$P+Q\big(f(t)\big)^2 = P+Q\sum_{k=0}^{\infty}a_k t^{k\alpha}\sum_{n=0}^{\infty}a_n t^{n\alpha} = \sum_{m=0}^{\infty}\left[P\delta_{m0}+Q\sum_{k=0}^{m}a_k a_{m-k}\right]t^{m\alpha}.$$

Thus the nonlinear Riccati equation (91) has a solution of the form (92) with the coefficients a_k defined to satisfy

$$\begin{gathered}
0 = P+Qa_0^2;\\
\tfrac{B(\alpha)}{1-\alpha}\sum_{k=1}^{m}a_k\left(\tfrac{-\alpha}{1-\alpha}\right)^{m-k}\frac{\Gamma(k\alpha+1)}{\Gamma(m\alpha+1)} = Q\sum_{k=0}^{m}a_k a_{m-k} \qquad \text{for all } m>0.
\end{gathered}$$

This identity can be used to find all the coefficients, by solving it for each value of m in turn. For $m=0$, we have $0=P+Qa_0^2$, so $a_0=\pm\sqrt{-P/Q}$. For $m>0$, we have

$$a_m = \frac{\frac{B(\alpha)}{1-\alpha}\sum_{k=1}^{m-1}a_k\left(\frac{-\alpha}{1-\alpha}\right)^{m-k}\frac{\Gamma(k\alpha+1)}{\Gamma(m\alpha+1)} - Q\sum_{k=1}^{m-1}a_k a_{m-k}}{2Qa_0-\frac{B(\alpha)}{1-\alpha}}.$$

Thus, the general solution of the form (92) to the Riccati equation (91) is

$$f(t) = \sqrt{\frac{-P}{Q}}+\sum_{m=1}^{\infty}\frac{\frac{B(\alpha)}{1-\alpha}\sum_{k=1}^{m-1}a_k\left(\frac{-\alpha}{1-\alpha}\right)^{m-k}\frac{\Gamma(k\alpha+1)}{\Gamma(m\alpha+1)} - Q\sum_{k=1}^{m-1}a_k a_{m-k}}{2Qa_0-\frac{B(\alpha)}{1-\alpha}}t^{m\alpha}.$$

This is an extension to the AB model of the results of [42] for series solutions of the Riccati equation. Note that this simple solution was only possible because we were able to use the new series formula (70) for AB derivatives. The fractional derivative on the left hand side of the ODE became a double series to match the double series on the right hand side, and we could solve the resulting identity very directly.

2.4.5 The semigroup property

The semigroup property for AB fractional differintegrals is *not* satisfied in general. For example, taking $B(\alpha)=1$ we find

$$ {}^{AB}_{\;\;0}I^{2/3}_t(t)=\left(\tfrac{1}{3}+\tfrac{2}{3}\,{}^{RL}_{\;\;0}I^{2/3}_t\right)t=\tfrac{1}{3}t+\tfrac{2}{3\Gamma(8/3)}t^{5/3}, $$

and yet

$$ \begin{aligned} {}^{AB}_{\;\;0}I^{1/3}_t\left({}^{AB}_{\;\;0}I^{1/3}_t(t)\right)&={}^{AB}_{\;\;0}I^{1/3}_t\left(\tfrac{2}{3}t+\tfrac{1}{3\Gamma(7/3)}t^{4/3}\right)\\ &=\tfrac{2}{3}\left(\tfrac{2}{3}t+\tfrac{1}{3\Gamma(7/3)}t^{4/3}\right)+\tfrac{1}{3\Gamma(7/3)}\left(\tfrac{2}{3}t^{4/3}+\tfrac{\Gamma(7/3)}{\Gamma(8/3)}t^{5/3}\right)\\ &=\tfrac{4}{9}t+\tfrac{4}{9\Gamma(7/3)}t^{4/3}+\tfrac{1}{3\Gamma(8/3)}t^{5/3}, \end{aligned} $$

two entirely different expressions.

Can we find conditions for when the semigroup property *does* hold?

Firstly, note that it will be sufficient to consider fractional *integrals* only. Any function which satisfies the semigroup property for ABR fractional derivatives generates one which satisfies it for AB fractional integrals, and vice versa. This is because

$$ {}^{ABR}_{\;\;\;0}I^{\alpha}_t\left({}^{ABR}_{\;\;\;0}I^{\beta}_t f(t)\right)=g(t)={}^{ABR}_{\;\;\;0}I^{\alpha+\beta}_t f(t) $$

is exactly equivalent to

$$ {}^{ABR}_{\;\;\;0}D^{\beta}_t\left({}^{ABR}_{\;\;\;0}D^{\alpha}_t g(t)\right)=f(t)={}^{ABR}_{\;\;\;0}D^{\alpha+\beta}_t g(t). $$

This is good to know, because the definition of AB fractional integrals is much simpler and easier to work with than that of ABR fractional derivatives.

The semigroup property for AB fractional integrals is equivalent to the following conditions. For ease of notation, we are using ${}^{AB}I$ to denote AB fractional integrals and just I to denote Riemann–Liouville fractional integrals.

$$ \begin{aligned} &{}^{AB}I^{\alpha}\left({}^{AB}I^{\beta}f(t)\right)={}^{AB}I^{\alpha+\beta}f(t)\\ \Leftrightarrow\;&\left(\tfrac{1-\alpha}{B(\alpha)}+\tfrac{\alpha}{B(\alpha)}I^{\alpha}\right)\left(\tfrac{1-\beta}{B(\beta)}+\tfrac{\beta}{B(\beta)}I^{\beta}\right)f(t)=\left(\tfrac{1-\alpha-\beta}{B(\alpha+\beta)}+\tfrac{\alpha+\beta}{B(\alpha+\beta)}I^{\alpha+\beta}\right)f(t)\\ \Leftrightarrow\;&\tfrac{(1-\alpha)(1-\beta)}{B(\alpha)B(\beta)}f+\tfrac{\alpha(1-\beta)}{B(\alpha)B(\beta)}I^{\alpha}f+\tfrac{\beta(1-\alpha)}{B(\alpha)B(\beta)}I^{\beta}f+\tfrac{\alpha\beta}{B(\alpha)B(\beta)}I^{\alpha+\beta}f\\ &\qquad\qquad\qquad\qquad=\tfrac{1-\alpha-\beta}{B(\alpha+\beta)}f+\tfrac{\alpha+\beta}{B(\alpha+\beta)}I^{\alpha+\beta}f\\ \Leftrightarrow\;&\left[\tfrac{\alpha\beta}{B(\alpha)B(\beta)}-\tfrac{\alpha+\beta}{B(\alpha+\beta)}\right]I^{\alpha+\beta}f+\tfrac{\alpha(1-\beta)}{B(\alpha)B(\beta)}I^{\alpha}f+\tfrac{\beta(1-\alpha)}{B(\alpha)B(\beta)}I^{\beta}f\\ &\qquad\qquad\qquad\qquad+\left[\tfrac{(1-\alpha)(1-\beta)}{B(\alpha)B(\beta)}-\tfrac{1-\alpha-\beta}{B(\alpha+\beta)}\right]f=0. \end{aligned} \tag{93} $$

Thus we have a Riemann–Liouville fractional integral equation in f which must be satisfied in order for the AB fractional integrals of f to have the semigroup property. If we assume for simplicity that the normalisation function B satisfies its own semigroup property $B(\alpha)B(\beta) = B(\alpha+\beta)$, then we can simplify the condition (93) as follows:

$$\begin{aligned}
(93) \Leftrightarrow &\big(\alpha\beta-\alpha-\beta\big)I^{\alpha+\beta}f + \big(\alpha-\alpha\beta\big)I^{\alpha}f + \big(\beta-\beta\alpha\big)I^{\beta}f + \alpha\beta f = 0 \\
\Leftrightarrow &\,\alpha\beta\Big(I^{\alpha+\beta}f - I^{\alpha}f - I^{\beta}f + f\Big) + \alpha\big(I^{\alpha}f - I^{\alpha+\beta}f\big) + \beta\big(I^{\beta}f - I^{\alpha+\beta}f\big) = 0 \\
\Leftrightarrow &\,\alpha\beta\big(I^{\alpha}-1\big)\big(I^{\beta}-1\big)f - \alpha I^{\alpha}\big(I^{\beta}-1\big)f - \beta I^{\beta}\big(I^{\alpha}-1\big)f = 0 \\
\Leftrightarrow &\Big[\alpha\big(I^{\beta}-1\big) - I^{\beta}\Big]\Big[\beta\big(I^{\alpha}-1\big) - I^{\alpha}\Big]f - I^{\alpha+\beta}f = 0 \\
\Leftrightarrow &\Big[(\alpha-1)I^{\beta}-\alpha\Big]\Big[(\beta-1)I^{\alpha}-\beta\Big]f = I^{\alpha+\beta}f.
\end{aligned}$$

Using the composition properties of Riemann–Liouville fractional differintegrals, we can derive a necessary condition for the semigroup property in the form of a Riemann–Liouville fractional *differential* equation by applying $D^{\alpha+\beta} = {}^{RL}_{\;0}D^{\alpha+\beta}_{t}$ to (93):

$$\begin{aligned}
(93) \Rightarrow \left[\frac{(1-\alpha)(1-\beta)}{B(\alpha)B(\beta)} - \frac{1-\alpha-\beta}{B(\alpha+\beta)}\right]D^{\alpha+\beta}f + \frac{\alpha(1-\beta)}{B(\alpha)B(\beta)}D^{\beta}f + \frac{\beta(1-\alpha)}{B(\alpha)B(\beta)}D^{\alpha}f& \\
+\left[\frac{\alpha\beta}{B(\alpha)B(\beta)} - \frac{\alpha+\beta}{B(\alpha+\beta)}\right]f = 0& \quad (94)
\end{aligned}$$

Assuming that $B(\alpha)B(\beta) = B(\alpha+\beta)$, we can again rewrite this condition in a more elegant form:

$$(94) \Leftrightarrow \alpha\beta D^{\alpha+\beta}f + \alpha(1-\beta)D^{\beta}f + \beta(1-\alpha)D^{\alpha}f + (\alpha\beta-\alpha-\beta)f = 0.$$

Using the methods described in [105], we can solve this fractional ODE for rational α, β by finding the roots of the indicial polynomial

$$\begin{aligned}
P(x) &= \alpha\beta x^{\alpha+\beta} + \alpha(1-\beta)x^{\beta} + \beta(1-\alpha)x^{\alpha} + (\alpha\beta-\alpha-\beta) \\
&= (\beta x^{\alpha} - \beta + 1)(\alpha x^{\beta} - \alpha + 1) - 1.
\end{aligned}$$

This equation is not going to have neat solutions in general, but in the case where $\alpha = \beta$ things become easier. In this case, our assumption on B from earlier becomes $B(\alpha)^2 = B(2\alpha)$ and the indicial polynomial is

$$\begin{aligned}
P(x) &= \alpha^2 x^{2\alpha} + 2\alpha(1-\alpha)x^{\alpha} + (\alpha^2 - 2\alpha) = \alpha^2\big(x^{2\alpha} + \tfrac{2-2\alpha}{\alpha}x^{\alpha} + \tfrac{\alpha-2}{\alpha}\big) \\
&= \alpha^2\big(x-1\big)\big(x - \tfrac{\alpha-2}{\alpha}\big).
\end{aligned}$$

Here the indicial polynomial is relatively easy to solve, and we can use its roots to construct a solution f to the fractional ODE (94) in the form of a linear combination of incomplete gamma functions.

Example 2.4.12. For example, let us consider the simplest case, that in which $\alpha = \beta = \frac{1}{q}$ for some natural number $q > 2$. Here we have

$$P'(x) = 2\alpha(\alpha x - \alpha + 1), P'(1) = 2\alpha, P'\left(\tfrac{\alpha-2}{\alpha}\right) = -2\alpha$$

and by [105] the solution to the fractional ODE (94) is

$$f(t) = \frac{1}{2\alpha}\sum_{k=0}^{q-1} E_t(-k\alpha, 1) - \frac{1}{2\alpha}\sum_{k=0}^{q-1} \left(\tfrac{\alpha-2}{\alpha}\right)^{q-k-1} E_t\left(-k\alpha, \left(\tfrac{\alpha-2}{\alpha}\right)^q\right) \tag{95}$$

where E in this case is the relative of the Mittag-Leffler function defined by $E_t(\nu, a) = z^\nu \sum_{n=0}^{\infty} \frac{(at)^n}{\Gamma(\nu+n+1)}$.

So the function f defined by (95) is the only function satisfying the semigroup property

$${}^{ABR}_{0}I^{\alpha}_{t}\left({}^{ABR}_{0}I^{\alpha}_{t} f(t)\right) = {}^{ABR}_{0}I^{2\alpha}_{t} f(t)$$

where $\alpha = \frac{1}{q}$, $q > 2$ is a natural number, and B satisfies $B(\alpha)^2 = B(2\alpha)$. □

The above is just one example, but we can see that in general the semigroup property for AB integrals of a function f is equivalent to a Riemann–Liouville ODE in f, which is only going to be satisfied by a small class of functions. So not only is the semigroup property not universally valid, it is not even valid given some initial or decay conditions: it is essentially *never* valid, except for a very specific family of functions f.

2.4.6 The product and chain rules

It is possible to establish a **product rule** for AB fractional derivatives using the series formula from §2.4.2. Our starting point is the already-known result of Lemma 1.1.8 for RL fractional differintegrals. Note that this covers fractional integrals as well as derivatives: for $\alpha > 0$ we have

$${}^{RL}_{a}I^{\alpha}_{t}\big(u(t)v(t)\big) = \sum_{n=0}^{\infty} \binom{-\alpha}{n} {}^{RL}_{a}I^{\alpha+n}_{t} u(t) \frac{\mathrm{d}^n v}{\mathrm{d}t^n}. \tag{96}$$

Thus it can be used to prove a corresponding identity for ABR fractional derivatives.

Theorem 2.4.13. *Let $\alpha \in (0,1)$ and $[a,b] \subset \mathbb{R}$. Let u and v be such that u, v, and uv are all in $L^1[a,b]$ and all of the form $x^\lambda \eta(x)$ with* $\mathrm{Re}(\lambda) > -1$ *and η analytic on $[a,b]$.*

Then

$$
{}^{ABR}_{\ a}D^{\alpha}_{t}\big(u(t)v(t)\big)=\sum_{m=0}^{\infty}\frac{\mathrm{d}^m v}{\mathrm{d}t^m}\left[\frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\Big(\frac{-\alpha}{1-\alpha}\Big)^{n}\binom{-n\alpha}{m}\,{}^{RL}_{\ a}I^{\alpha n+m}_{t}u(t)\right]. \tag{97}
$$

Proof. We start from the series formula (65) for ABR derivatives, and substitute in the fractional product rule (96) for RL integrals to get a double series:

$$
\begin{aligned}
{}^{ABR}_{\ a}D^{\alpha}_{t}\big(u(t)v(t)\big)&=\frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\Big(\frac{-\alpha}{1-\alpha}\Big)^{n}\,{}^{RL}_{\ a}I^{\alpha n}_{t}\big(u(t)v(t)\big)\\
&=\frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\Big(\frac{-\alpha}{1-\alpha}\Big)^{n}\binom{-n\alpha}{m}\,{}^{RL}_{\ a}I^{\alpha n+m}_{t}u(t)\frac{\mathrm{d}^m v}{\mathrm{d}t^m}\\
&=\sum_{m=0}^{\infty}\frac{\mathrm{d}^m v}{\mathrm{d}t^m}\left[\frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\Big(\frac{-\alpha}{1-\alpha}\Big)^{n}\binom{-n\alpha}{m}\,{}^{RL}_{\ a}I^{\alpha n+m}_{t}u(t)\right].
\end{aligned}
$$

Note that when $m=0$, the term in square brackets is exactly ${}^{ABR}_{\ a}D^{\alpha}_{t}u(t)$.

In order to prove rigorously that the double series here converges, we shall express it as the sum of a finite series and a remainder term, and then prove that the remainder term tends to zero. The following formula is equation (2.199) in [118]:

$$
{}^{RL}_{\ a}D^{\alpha}_{t}\big(u(t)v(t)\big)=\sum_{n=0}^{N}\binom{\alpha}{n}\,{}^{RL}_{\ a}D^{\alpha-n}_{t}u(t)\,{}^{RL}_{\ a}D^{n}_{t}v(t)-R^{\alpha}_{N}(t), \tag{98}
$$

valid for $\alpha\in\mathbb{R}, n\geq\alpha+1, u\in C[a,t], v\in C^{N+1}[a,t]$, where the remainder term $R^{\alpha}_{N}(t)$ is defined by

$$
R^{\alpha}_{N}(t)=\frac{1}{N!\Gamma(-\alpha)}\int_a^t(t-\tau)^{-\alpha-1}u(\tau)\left[\int_\tau^t v^{(N+1)}(\xi)(\tau-\xi)^N\,\mathrm{d}\xi\right]\mathrm{d}\tau.
$$

Applying (98) with α replaced by $-n\alpha$ for $n\in\mathbb{N}$, we find

$$
\begin{aligned}
{}^{ABR}_{\ a}D^{\alpha}_{t}\big(u(t)v(t)\big)&=\frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\Big(\frac{-\alpha}{1-\alpha}\Big)^{n}\,{}^{RL}_{\ a}I^{\alpha n}_{t}\big(u(t)v(t)\big)\\
&=\frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\left[\sum_{m=0}^{N}\big(\tfrac{-\alpha}{1-\alpha}\big)^{n}\tbinom{-n\alpha}{m}\,{}^{RL}_{\ a}D^{-n\alpha-m}_{t}u(t)\,{}^{RL}_{\ a}D^{m}_{t}v(t)-R^{-n\alpha}_{N}(t)\right]\\
&=\sum_{m=0}^{N}\frac{\mathrm{d}^m v}{\mathrm{d}t^m}\left[\frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\big(\tfrac{-\alpha}{1-\alpha}\big)^{n}\tbinom{-n\alpha}{m}\,{}^{RL}_{\ a}I^{\alpha n+m}_{t}u(t)\right]\\
&\qquad\qquad-\frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}R^{-n\alpha}_{N}(t),
\end{aligned} \tag{99}
$$

where the interchange of summations is valid because by Theorem 2.4.4 we know the sum over n converges locally uniformly whenever the relevant ABR derivative is well-defined. So to establish the convergence of the outer series in (97), it will suffice to prove that

$$\lim_{N\to\infty}\sum_{n=0}^{\infty}R_N^{-n\alpha}(t)=0$$

for $v\in C^{\infty}[a,t]$. And this can be proven by almost exactly the same argument as in [118]. Equation (2.201) from there gives

$$R_N^{-n\alpha}(t)=\frac{(-1)^N(t-a)^{N+n\alpha+1}}{N!\Gamma(n\alpha)}\int_0^1\int_0^1 u\big(a+\eta(t-a)\big)v^{(N+1)}\big(a+(t-a)(\zeta+\eta-\zeta\eta)\big)\,\mathrm{d}\eta\,\mathrm{d}\zeta,$$

so

$$\sum_{n=0}^{\infty}R_N^{-n\alpha}(t)=\frac{(-1)^N(t-a)^{N+2}}{N!}\frac{\mathrm{d}}{\mathrm{d}t}\Big(E_\alpha\big((t-a)^\alpha\big)\Big)\times \\ \int_0^1\int_0^1 u\big(a+\eta(t-a)\big)v^{(N+1)}\big(a+(t-a)(\zeta+\eta-\zeta\eta)\big)\,\mathrm{d}\eta\,\mathrm{d}\zeta,$$

which converges to 0 as $N\to\infty$, as required. Note that in order to derive the final expression, we used the fact that

$$\sum_{n=0}^{\infty}\frac{t^{-n\alpha}}{\Gamma(n\alpha)}=-\sum_{n=0}^{\infty}\frac{-n\alpha t^{-n\alpha}}{\Gamma(n\alpha+1)}=-t\frac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_{n=0}^{\infty}\frac{t^{-n\alpha}}{\Gamma(n\alpha+1)}\Big)=-t\frac{\mathrm{d}}{\mathrm{d}t}E_\alpha\big(t^{-\alpha}\big).$$

Thus (97) is valid as a generalisation of the Leibniz rule to ABR fractional derivatives. □

Example 2.4.14. As an example to verify this new result, let us take $u(t)=t^2$ and $v(t)=t$ with $a=0$. Then the right-hand side of (97) becomes

$$\begin{aligned}
&\sum_{m=0}^{1}\frac{\mathrm{d}^m}{\mathrm{d}t^m}(t)\left[\frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\Big(\frac{-\alpha}{1-\alpha}\Big)^n\binom{-n\alpha}{m}\,{}^{RL}_{0}I^{\alpha n+m}_{t}(t^2)\right]\\
&=t\left[\frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\Big(\frac{-\alpha}{1-\alpha}\Big)^n\frac{2}{\Gamma(3+\alpha n)}t^{2+\alpha n}\right]\\
&\qquad+\left[\frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\Big(\frac{-\alpha}{1-\alpha}\Big)^n(-n\alpha)\frac{2}{\Gamma(4+\alpha n)}t^{3+\alpha n}\right]\\
&=\frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\Big(\frac{-\alpha}{1-\alpha}\Big)^n\left[\frac{2}{\Gamma(3+\alpha n)}-\frac{2n\alpha}{\Gamma(4+\alpha n)}\right]t^{3+\alpha n}\\
&=\frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\Big(\frac{-\alpha}{1-\alpha}\Big)^n\frac{6}{\Gamma(4+\alpha n)}t^{3+\alpha n}=\frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\Big(\frac{-\alpha}{1-\alpha}\Big)^n\,{}^{RL}_{0}I^{\alpha n}_{t}(t^3)\\
&={}^{ABR}_{0}D^{\alpha}_{t}(t^3),
\end{aligned}$$

exactly as expected. $\square$

The product rule is an extremely important idea to examine in any new model of fractional calculus. As was pointed out by Tarasov [136], the Leibniz rule plays a crucial role in fractional calculus and its applications, to the extent that it has been proposed as a test for the validity of a given model.

Furthermore, having an analogue of the product rule enables us to greatly extend the number of functions whose fractional derivatives can be calculated. We now have a way to compute AB derivatives of anything which can be expressed as a product of two or more functions whose AB derivatives are already known – thus expanding the space of functions on which we can easily do calculations. The expression (97) is admittedly cumbersome, but this is a common problem with fractional generalisations of results from calculus: we see it also in the classical result (7).

Similarly, we can prove a **chain rule** for AB fractional derivatives by starting from the identity (8) from the RL model. Again this result covers fractional integrals as well as derivatives, so we can use it to get a corresponding identity for ABR fractional derivatives. For $\alpha \in (0,1)$ and any f, g as above such that $f(g(t))$ is an L^1 function,

$$
\begin{aligned}
{}^{ABR}_{\ \ a}D^{\alpha}_{t} f(g(t)) &= \frac{B(\alpha)}{1-\alpha}\sum_{m=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{m} {}^{RL}_{\ \ a}I^{\alpha m}_{t} f(g(t)) \\
&= \frac{B(\alpha)}{1-\alpha}\sum_{m=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{m}\left[\frac{(t-a)^{m\alpha}}{\Gamma(m\alpha+1)}f(g(t)) + \sum_{n=1}^{\infty}\binom{-m\alpha}{n}\frac{(t-a)^{n+m\alpha}}{\Gamma(n+m\alpha+1)}\right. \\
&\qquad\qquad \left.\sum_{k=1}^{n}\frac{\mathrm{d}^k f(g(t))}{\mathrm{d}g(t)^k}\sum_{(P_1,\dots,P_n)}\left[\prod_{i=1}^{n}\tfrac{i}{P_i!(i!)^{P_i}}\left(\tfrac{\mathrm{d}^i g(t)}{\mathrm{d}t^i}\right)^{P_i}\right]\right] \\
&= \frac{B(\alpha)}{1-\alpha}\left[E_{\alpha}\left(\tfrac{-\alpha}{1-\alpha}(t-a)^{\alpha}\right)f(g(t))\right. \\
&\qquad \left. + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(\tfrac{-\alpha}{1-\alpha}\right)^{m}\binom{-m\alpha}{n}\frac{(t-a)^{n+m\alpha}}{\Gamma(n+m\alpha+1)}\sum_{k=1}^{n}(\dots)\right],
\end{aligned}
\tag{100}
$$

where the series in (8) are convergent by the proof in [112] and the outer series in (100) is locally uniformly convergent by Theorem 2.4.4, therefore the sums can be interchanged and the result is a well-defined convergent series.

Example 2.4.15. As an example to verify this new identity, let us take $a = 0$ and $g(t) = e^t$ and $f(t) = t^2$ so that $f(g(t)) = e^{2t}$. Then the kth derivative of $f(g(t))$ with respect to $g(t)$ is $2e^t$ if $k = 1$, 2 if $k = 2$, and 0 if $k > 2$. For $k = 1$, we must have $(P_1, \dots, P_n) = (0, \dots, 0, 1)$ and therefore

$$
\prod_{i=1}^{n}\tfrac{i}{P_i!(i!)^{P_i}}\left(\tfrac{\mathrm{d}^i g(t)}{\mathrm{d}t^i}\right)^{P_i} = \frac{\prod_{i=1}^{n} i}{1!(n!)^1}\left(e^t\right)^1 = e^t.
$$

For $k=2$, we must have either $P_j = P_{n-j} = 1$ for some $j \neq \frac{n}{2}$ and all other $P_i = 0$ or (if n is even) $P_{n/2} = 2$ and all other $P_i = 0$. In the first case,

$$\prod_{i=1}^{n} \frac{i}{P_i!(i!)^{P_i}} \left(\frac{\mathrm{d}^i g(t)}{\mathrm{d}t^i}\right)^{P_i} = \frac{\prod_{i=1}^{n} i}{(1!)^2 j!(n-j)!} \left(e^t\right)^1 \left(e^t\right)^1 = \binom{n}{j} e^{2t}$$

while in the second case,

$$\prod_{i=1}^{n} \frac{i}{P_i!(i!)^{P_i}} \left(\frac{\mathrm{d}^i g(t)}{\mathrm{d}t^i}\right)^{P_i} = \frac{\prod_{i=1}^{n} i}{2!(\frac{n}{2}!)^2} \left(e^t\right)^2 = \frac{1}{2}\binom{n}{n/2} e^{2t}.$$

So the right-hand side of (100) becomes

$$\frac{B(\alpha)}{1-\alpha}\Bigg[E_\alpha\left(\tfrac{-\alpha}{1-\alpha}t^\alpha\right)e^{2t} \\ + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(\tfrac{-\alpha}{1-\alpha}\right)^m \tbinom{-m\alpha}{n} \frac{t^{n+m\alpha}}{\Gamma(n+m\alpha+1)} \left((2e^t)e^t + (2)\sum_{j=1}^{\lfloor n/2 \rfloor} \tbinom{n}{j} e^{2t} + \tbinom{n}{n/2} e^{2t}\right)\Bigg],$$

where the last term is present only if n is even. This simplifies to

$$\begin{aligned}
&= \frac{B(\alpha)}{1-\alpha}\left[E_\alpha\left(\tfrac{-\alpha}{1-\alpha}t^\alpha\right)e^{2t} + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(\tfrac{-\alpha}{1-\alpha}\right)^m \tbinom{-m\alpha}{n} \frac{t^{n+m\alpha}}{\Gamma(n+m\alpha+1)} \sum_{j=0}^{n} \tbinom{n}{j} e^{2t}\right] \\
&= \frac{B(\alpha)}{1-\alpha}\left[\sum_{m=0}^{\infty}\left(\tfrac{-\alpha}{1-\alpha}\right)^m \frac{t^{\alpha m}}{\Gamma(\alpha m+1)} e^{2t} + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(\tfrac{-\alpha}{1-\alpha}\right)^m \tbinom{-m\alpha}{n} \frac{t^{n+m\alpha}}{\Gamma(n+m\alpha+1)} 2^n e^{2t}\right] \\
&= \frac{B(\alpha)}{1-\alpha}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\tfrac{-\alpha}{1-\alpha}\right)^m \tbinom{-m\alpha}{n} \frac{t^{n+m\alpha}}{\Gamma(n+m\alpha+1)} 2^n e^{2t},
\end{aligned}$$

which is exactly the formula for ${}^{ABR}_{0}D^\alpha_t(e^{2t})$, since the Riemann–Liouville integrals of the exponential function are given by ${}^{RL}_{0}I^{m\alpha}_t(e^{2t}) = \sum_{n=0}^{\infty} \frac{2^n t^{m\alpha+n}}{\Gamma(m\alpha+n+1)}$. □

Once again, being able to use the chain rule enables a big extension to the class of functions with easily computable fractional derivatives. With the results of (97) and (100), we can now explicitly compute AB derivatives of anything which can be derived by multiplication and composition from two or more functions with already known AB derivatives. So once more we have expanded the space of functions on which it is easy to perform calculations with AB derivatives.

2.4.7 The mean value theorem and Taylor's theorem

Versions of the mean value theorem and Taylor's theorem are already known in the standard Riemann–Liouville [139] and Caputo [107] models, and versions of the mean value theorem for fractional *difference* operators have been proved in both the Caputo–Fabrizio model [3] and the AB model [2], but a fractional mean value theorem in the continuous AB model had not been established before the current work.

The following analogue of the mean value theorem for ABC fractional derivatives follows from our Newton–Leibniz formula (72).

Theorem 2.4.16 (AB mean value theorem)**.** *Let $0 < \alpha < 1$, $a < b$ in $\mathbb{R}$, and $f : [a, b] \to \mathbb{R}$ differentiable such that $f' \in L^1[a, b]$ and ${}^{ABC}_{a}D^{\alpha} f \in C[a, b]$. Then for any $t \in [a, b]$, there exists $\xi \in [a, t]$ such that*

$$f(t) = f(a) + \frac{1-\alpha}{B(\alpha)} \, {}^{ABC}_{a}D^{\alpha}_{t} f(t) + \frac{(t-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)} \, {}^{ABC}_{a}D^{\alpha}_{\xi} f(\xi). \tag{101}$$

Proof. By Theorem 2.4.9, we have:

$$\begin{aligned} f(t) - f(a) &= {}^{AB}_{a}I^{\alpha}_{t} \left({}^{ABC}_{a}D^{\alpha}_{t} f(t) \right) \\ &= \frac{1-\alpha}{B(\alpha)} \, {}^{ABC}_{a}D^{\alpha}_{t} f(t) + \frac{\alpha}{B(\alpha)} \, {}^{RL}_{a}I^{\alpha}_{t} \left({}^{ABC}_{a}D^{\alpha}_{t} f(t) \right) \\ &= \frac{1-\alpha}{B(\alpha)} \, {}^{ABC}_{a}D^{\alpha}_{t} f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} \left({}^{ABC}_{a}D^{\alpha}_{x} f(x) \right) \mathrm{d}x. \end{aligned}$$

Now by the integral mean value theorem, since ${}^{ABC}_{a}D^{\alpha}_{x} f(x)$ is continuous and $(t-x)^{\alpha-1}$ is integrable and positive, there exists $\xi \in (a, t)$ such that

$$\begin{aligned} f(t) - f(a) &= \frac{1-\alpha}{B(\alpha)} \, {}^{ABC}_{a}D^{\alpha}_{t} f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \, {}^{ABC}_{a}D^{\alpha}_{\xi} f(\xi) \int_a^t (t-x)^{\alpha-1} \mathrm{d}x \\ &= \frac{1-\alpha}{B(\alpha)} \, {}^{ABC}_{a}D^{\alpha}_{t} f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \, {}^{ABC}_{a}D^{\alpha}_{\xi} f(\xi) \frac{(t-a)^{\alpha}}{\alpha}, \end{aligned}$$

as required. ◻

For interest's sake we also include the following corollary, another form of the ABC fractional mean value theorem in terms of an inequality.

Corollary 2.4.17. *With all notations and assumptions as in Theorem 2.4.16, if f is monotonic (increasing or decreasing), then*

$$f(t) \geq f(a) + \left[1 + E_{\alpha} \left(\frac{-\alpha}{1-\alpha} (t-a)^{\alpha} \right) \right]^{-1} \frac{(t-a)^{\alpha}}{B(\alpha)\Gamma(\alpha)} \, {}^{ABC}_{a}D^{\alpha}_{\xi} f(\xi) \tag{102}$$

for some $\xi \in (a, t)$.

Proof. We shall start from the equation (101) to derive this inequality. Firstly, using the integral mean value theorem again, we can write the ABC derivative as

$$\begin{aligned}
\frac{1-\alpha}{B(\alpha)}\,{}^{ABC}_{\;\;a}D^{\alpha}_{t}f(t) &= \int_a^t f'(x)E_\alpha\left(\frac{-\alpha}{1-\alpha}(t-x)^\alpha\right)\mathrm{d}x \\
&= E_\alpha\left(\frac{-\alpha}{1-\alpha}(t-c)^\alpha\right)\int_a^t f'(x)\mathrm{d}x \\
&= \big(f(t)-f(a)\big)E_\alpha\left(\frac{-\alpha}{1-\alpha}(t-c)^\alpha\right)
\end{aligned}$$

for some $c \in (a,t)$, since E_α is continuous and f' is integrable and has constant sign. We substitute this into (101) to find

$$f(t)-f(a) = \big(f(t)-f(a)\big)E_\alpha\left(\frac{-\alpha}{1-\alpha}(t-c)^\alpha\right) + \frac{(t-a)^\alpha}{B(\alpha)\Gamma(\alpha)}\,{}^{ABC}_{\;\;a}D^{\alpha}_{\xi}f(\xi)$$

and therefore

$$f(t) = f(a) + \left[1-E_\alpha\left(\frac{-\alpha}{1-\alpha}(t-c)^\alpha\right)\right]^{-1}\frac{(t-a)^\alpha}{B(\alpha)\Gamma(\alpha)}\,{}^{ABC}_{\;\;a}D^{\alpha}_{\xi}f(\xi).$$

Since the Mittag-Leffler function on a negative argument is completely monotone [120], the result follows. □

Before starting to prove analogues of Taylor's theorem for fractional AB derivatives, we first establish the following lemma.

Lemma 2.4.18. *If $\alpha \in (0,1)$ and $a < b$ in $\mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ is a differentiable function such that f' and all functions of the form $\left({}^{ABC}_{\;\;a}D^{\alpha}_{t}\right)^m f(t), m \in \mathbb{N}$, are L^1 functions, then*

$$\begin{aligned}
&\left({}^{AB}_{\;\;a}I^{\alpha}_{t}\right)^m\left({}^{ABC}_{\;\;a}D^{\alpha}_{t}\right)^m f(t) - \left({}^{AB}_{\;\;a}I^{\alpha}_{t}\right)^{m+1}\left({}^{ABC}_{\;\;a}D^{\alpha}_{t}\right)^{m+1} f(t) \\
&\qquad = \frac{\left({}^{ABC}_{\;\;a}D^{\alpha}_{t}\right)^m f(a)}{B(\alpha)^m}\sum_{k=0}^{m}\frac{\binom{m}{k}(1-\alpha)^{m-k}\alpha^k}{\Gamma(k\alpha+1)}(t-a)^{k\alpha} \qquad (103)
\end{aligned}$$

for all $m \in \mathbb{N}$.

Proof. By Theorem 2.4.9, we know that

$$\left(1-{}^{AB}_{\;\;a}I^{\alpha}_{t}\,{}^{ABC}_{\;\;a}D^{\alpha}_{t}\right)f(t) = f(a). \qquad (104)$$

So the left-hand side of the equation (103) can be written as follows, where we denote

${}^{AB}_{a}I^{\alpha}_{t}$ and ${}^{ABC}_{a}D^{\alpha}_{t}$ by simply I^{α} and D^{α} respectively for ease of notation:

$$\begin{aligned}
&\left(I^{\alpha}\right)^{m}\left(D^{\alpha}\right)^{m}f(t)-\left(I^{\alpha}\right)^{m+1}\left(D^{\alpha}\right)^{m+1}f(t)\\
&\qquad=\left(I^{\alpha}\right)^{m}\left(D^{\alpha}\right)^{m}f(t)-\left(I^{\alpha}\right)^{m}\left(I^{\alpha}D^{\alpha}\right)\left(D^{\alpha}\right)^{m}f(t)\\
&\qquad=\left(I^{\alpha}\right)^{m}\left(1-I^{\alpha}D^{\alpha}\right)\left(D^{\alpha}\right)^{m}f(t)\\
&\qquad=\left(I^{\alpha}\right)^{m}\left(\left(D^{\alpha}\right)^{m}f(a)\right),
\end{aligned}$$

where for the last step we used the identity (104). Denoting the constant $\left(D^{\alpha}\right)^{m}f(a)$ by A, we have

$$\begin{aligned}
\left({}^{AB}_{a}I^{\alpha}_{t}\right)^{m}(A)&=\left(\frac{1-\alpha}{B(\alpha)}+\frac{\alpha}{B(\alpha)}\,{}^{RL}_{a}I^{\alpha}_{t}\right)^{m}(A)\\
&=\frac{1}{B(\alpha)^{m}}\sum_{k=0}^{m}\binom{m}{k}(1-\alpha)^{m-k}\alpha^{k}\,{}^{RL}_{a}I^{k\alpha}_{t}(A)\\
&=\frac{A}{B(\alpha)^{m}}\sum_{k=0}^{m}\binom{m}{k}(1-\alpha)^{m-k}\alpha^{k}\frac{(t-a)^{k\alpha}}{\Gamma(k\alpha+1)},
\end{aligned}$$

as required. □

Now we are in a position to prove the following result, our first analogue of Taylor's theorem for fractional derivatives in the ABC model.

Theorem 2.4.19 (AB Taylor series about $t=a$). *If $\alpha\in(0,1)$ and $n\in\mathbb{N}$ and $a<b$ in $\mathbb{R}$ and $f:[a,b]\to\mathbb{R}$ is a differentiable function such that f' and all functions of the form $\left({}^{ABC}_{a}D^{\alpha}_{t}\right)^{m}f(t), m\in\mathbb{N}$, are L^{1} functions, then for all $t\in[a,b]$,*

$$f(t)=\sum_{m=0}^{n}S_{\alpha,m}(t-a)\left({}^{ABC}_{a}D^{\alpha}_{t}\right)^{m}f(a)+S_{\alpha,n+1}(t-a)\left({}^{ABC}_{a}D^{\alpha}_{t}\right)^{n+1}f(\xi)\tag{105}$$

for some $\xi\in(a,t)$, where the function S is defined by

$$S_{\alpha,m}(x)\coloneqq\sum_{k=0}^{m}\frac{\binom{m}{k}(1-\alpha)^{m-k}\alpha^{k}}{B(\alpha)^{m}\Gamma(k\alpha+1)}x^{k\alpha}.\tag{106}$$

Proof. The result of Lemma 2.4.18 can be rewritten as

$$\begin{aligned}
&\left({}^{AB}_{a}I^{\alpha}_{t}\right)^{m}\left({}^{ABC}_{a}D^{\alpha}_{t}\right)^{m}f(t)-\left({}^{AB}_{a}I^{\alpha}_{t}\right)^{m+1}\left({}^{ABC}_{a}D^{\alpha}_{t}\right)^{m+1}f(t)\\
&\qquad\qquad=S_{\alpha,m}(t-a)\left({}^{ABC}_{a}D^{\alpha}_{t}\right)^{m}f(a),
\end{aligned}$$

valid for any $m \in \mathbb{N}$. Summing this identity over m to form a telescoping series, we get

$$f(t) - \left({}^{AB}_{a}I^{\alpha}_{t}\right)^{n+1} \left({}^{ABC}_{a}D^{\alpha}_{t}\right)^{n+1} f(t) = \sum_{m=0}^{n} S_{\alpha,m}(t-a) \left({}^{ABC}_{a}D^{\alpha}_{t}\right)^{m} f(a).$$

Thus it will suffice to prove that

$$\left({}^{AB}_{a}I^{\alpha}_{t}\right)^{n+1} \left({}^{ABC}_{a}D^{\alpha}_{t}\right)^{n+1} f(t) = S_{\alpha,n+1}(t-a) \left({}^{ABC}_{a}D^{\alpha}_{t}\right)^{n+1} f(\xi). \tag{107}$$

To establish (107), we use the mean value theorem for integrals once again, this time with one of the 'functions' involved being actually a distribution written in terms of the Dirac delta.

$$\begin{aligned}
&\left({}^{AB}_{a}I^{\alpha}_{t}\right)^{n+1} \left({}^{ABC}_{a}D^{\alpha}_{t}\right)^{n+1} f(t) \\
=&\left(\frac{1-\alpha}{B(\alpha)} + \frac{\alpha}{B(\alpha)}\,{}^{RL}_{a}I^{\alpha}_{t}\right)^{n+1} \left({}^{ABC}_{a}D^{\alpha}_{t}\right)^{n+1} f(t) \\
=&\sum_{k=0}^{n+1} \frac{\binom{n+1}{k}(1-\alpha)^{n+1-k}\alpha^k}{B(\alpha)^{n+1}}\,{}^{RL}_{a}I^{k\alpha}_{t}\left(\left({}^{ABC}_{a}D^{\alpha}_{t}\right)^{n+1} f(t)\right) \\
=&\left(\frac{1-\alpha}{B(\alpha)}\right)^{n+1} \left({}^{ABC}_{a}D^{\alpha}_{t}\right)^{n+1} f(t) \\
&\qquad + \sum_{k=1}^{n+1} \frac{\binom{n+1}{k}(1-\alpha)^{n+1-k}\alpha^k}{B(\alpha)^{n+1}\Gamma(k\alpha)} \int_a^t (t-x)^{k\alpha-1} \left({}^{ABC}_{a}D^{\alpha}_{x}\right)^{n+1} f(x)\mathrm{d}x \\
=&\int_a^t \Bigg[\left(\frac{1-\alpha}{B(\alpha)}\right)^{n+1} \delta(t-x) \\
&\qquad + \sum_{k=1}^{n+1} \frac{\binom{n+1}{k}(1-\alpha)^{n+1-k}\alpha^k}{B(\alpha)^{n+1}\Gamma(k\alpha)}(t-x)^{k\alpha-1}\Bigg] \left({}^{ABC}_{a}D^{\alpha}_{x}\right)^{n+1} f(x)\mathrm{d}x \\
=&\left({}^{ABC}_{a}D^{\alpha}_{\xi}\right)^{n+1} f(\xi) \int_a^t \Bigg[\left(\frac{1-\alpha}{B(\alpha)}\right)^{n+1} \delta(t-x) \\
&\qquad + \sum_{k=1}^{n+1} \frac{\binom{n+1}{k}(1-\alpha)^{n+1-k}\alpha^k}{B(\alpha)^{n+1}\Gamma(k\alpha)}(t-x)^{k\alpha-1}\Bigg]\mathrm{d}x \\
=&\left({}^{ABC}_{a}D^{\alpha}_{\xi}\right)^{n+1} f(\xi) \left[\left(\frac{1-\alpha}{B(\alpha)}\right)^{n+1} + \sum_{k=1}^{n+1} \frac{\binom{n+1}{k}(1-\alpha)^{n+1-k}\alpha^k}{B(\alpha)^{n+1}\Gamma(k\alpha+1)}(t-a)^{k\alpha}\right] \\
=&S_{\alpha,n+1}(t-a) \left({}^{ABC}_{a}D^{\alpha}_{\xi}\right)^{n+1} f(\xi),
\end{aligned}$$

as required. □

In order to get an infinite Taylor series expansion for a given function $f(t)$, it suffices to impose the following convergence condition on the remainder term:

$$S_{\alpha,n}(t-a)\left|\left|\left({}^{ABC}_{\ \ a}D^{\alpha}\right)^n f\right|\right| \to 0 \qquad \text{as } n\to\infty, \tag{108}$$

where the norm used is the uniform norm on $[a,t]$.

One disadvantage of Theorem 2.4.19 is that for many functions f, the ABC fractional derivative ${}^{ABC}_{\ \ a}D^{\alpha}_t f(t)$ evaluated at the starting point $t=a$ is zero. We can see this by considering the definition: since the ABC derivative is given by an integral from a to t, it will evaluate to zero given certain conditions on the behaviour of $f(t)$ near $t=a$. Thus, we present the following generalisation of Theorem 2.4.19, inspired by the work of [140].

Theorem 2.4.20 (AB Taylor series – general case)**.** *If $\alpha\in(0,1)$ and $n\in\mathbb{N}$ and $a<b$ in $\mathbb{R}$ and $f:[a,b]\to\mathbb{R}$ is a differentiable function such that f' and all functions of the form $\left({}^{ABC}_{\ \ a}D^{\alpha}_t\right)^m f(t), m\in\mathbb{N}$, are L^1 functions, then for all $c,t\in[a,b]$,*

$$f(t)=\sum_{m=0}^{n}\Delta_m\left({}^{ABC}_{\ \ a}D^{\alpha}_c\right)^m f(c)+R_{n+1}, \tag{109}$$

where the sequence of functions Δ_m is defined recursively by:

$$\Delta_{0,k}=S_{\alpha,k}(t-a),\quad \Delta_{m,k}=\Delta_{m-1,k}-\Delta_{m-1,m-1}S_{\alpha,k-m+1}(c-a) \tag{110}$$

and $\Delta_m=\Delta_{m.m}$, the functions $S_{\alpha,m}$ being defined by (106), and the remainder term R_{n+1} is a linear combination of terms of the form $\left({}^{ABC}_{\ \ a}D^{\alpha}_{\xi}\right)^{n+1} f(\xi)$ for $\xi\in(a,b)$.

Proof. We use the formula (105) from Theorem 2.4.19 as our starting point, and apply it multiple times in different ways to derive (109).

Replacing t by c in the equation (105), and replacing f by its ABC derivatives as appropriate, yields the following formulae for any fixed n (where we use the fact that

$S_{\alpha,0} = 1$):

$$
\begin{aligned}
f(a) &= f(c) - \sum_{m=1}^{n} S_{\alpha,m}(c-a) \left({}^{ABC}_{\ a}D^{\alpha}_{t}\right)^{m} f(a) \\
&\qquad\qquad\qquad - S_{\alpha,n+1}(c-a) \left({}^{ABC}_{\ a}D^{\alpha}_{t}\right)^{n+1} f(\xi_0), \\
{}^{ABC}_{\ a}D^{\alpha}_{t} f(a) &= {}^{ABC}_{\ a}D^{\alpha}_{c} f(c) - \sum_{m=1}^{n-1} S_{\alpha,m}(c-a) \left({}^{ABC}_{\ a}D^{\alpha}_{t}\right)^{m+1} f(a) \\
&\qquad\qquad\qquad - S_{\alpha,n}(c-a) \left({}^{ABC}_{\ a}D^{\alpha}_{t}\right)^{n+1} f(\xi_1), \\
\left({}^{ABC}_{\ a}D^{\alpha}_{t}\right)^{2} f(a) &= \left({}^{ABC}_{\ a}D^{\alpha}_{c}\right)^{2} f(c) - \sum_{m=0}^{n-2} S_{\alpha,m}(c-a) \left({}^{ABC}_{\ a}D^{\alpha}_{t}\right)^{m+2} f(a) \\
&\qquad\qquad\qquad - S_{\alpha,n-1}(c-a) \left({}^{ABC}_{\ a}D^{\alpha}_{t}\right)^{n+1} f(\xi_2), \\
&\ \ \vdots
\end{aligned}
$$

Substituting each of these equations in turn into (105) yields the following sequence of identities:

$$
\begin{aligned}
f(t) &= \Delta_{0,0} f(a) + \sum_{m=1}^{n} \Delta_{0,m} \left({}^{ABC}_{\ a}D^{\alpha}_{t}\right)^{m} f(a) + R_{0,n+1} \\
&= \Delta_{0,0} f(c) + \sum_{m=1}^{n} \left[\Delta_{0,m} - \Delta_{0,0} S_{\alpha,m}(c-a)\right] \left({}^{ABC}_{\ a}D^{\alpha}_{t}\right)^{m} f(a) + R_{1,n+1} \\
&= \Delta_{0,0} f(c) + \Delta_{1,1}\, {}^{ABC}_{\ a}D^{\alpha}_{t} f(a) + \sum_{m=2}^{n} \Delta_{1,m} \left({}^{ABC}_{\ a}D^{\alpha}_{t}\right)^{m} f(a) + R_{1,n+1} \\
&= \Delta_{0,0} f(c) + \Delta_{1,1}\, {}^{ABC}_{\ a}D^{\alpha}_{c} f(c) \\
&\qquad + \sum_{m=2}^{n} \left[\Delta_{1,m} - \Delta_{1,1} S_{\alpha,m+1}(c-a)\right] \left({}^{ABC}_{\ a}D^{\alpha}_{t}\right)^{m} f(a) + R_{2,n+1} \\
&= \Delta_{0,0} f(c) + \Delta_{1,1}\, {}^{ABC}_{\ a}D^{\alpha}_{c} f(c) + \Delta_{2,2} \left({}^{ABC}_{\ a}D^{\alpha}_{t}\right)^{2} f(a) \\
&\qquad\qquad\qquad + \sum_{m=3}^{n} \Delta_{2,m} \left({}^{ABC}_{\ a}D^{\alpha}_{t}\right)^{m} f(a) + R_{2,n+1} \\
&= \ldots,
\end{aligned}
$$

where the $\Delta_{k,m}$ are defined by (110) and the successive remainders are given by

$$
\begin{aligned}
R_{0,n+1} &= \Delta_{0,n+1} \left({}^{ABC}_{\ a}D^{\alpha}_{t}\right)^{n+1} f(\xi); \\
R_{k+1,n+1} &= R_{k,n+1} - \Delta_{k,k} S_{\alpha,n-k+1}(c-a) \left({}^{ABC}_{\ a}D^{\alpha}_{t}\right)^{n+1} f(\xi_k).
\end{aligned}
$$

After n iterations of this process, we arrive at the final result:

$$f(t) = \Delta_{0,0} f(c) + \Delta_{1,1} \, {}^{ABC}_{\ a}D^{\alpha}_{c} f(c) + \Delta_{2,2} \left({}^{ABC}_{\ a}D^{\alpha}_{t} \right)^2 f(c) \\ + \cdots + \Delta_{n,n} \left({}^{ABC}_{\ a}D^{\alpha}_{t} \right)^n f(a) + R_{n,n+1}.$$

Since $\Delta_m = \Delta_{m,m}$ by definition, and letting $R_{n+1} = R_{n,n+1}$, we discover equation (109) as required. Note that $\xi \in (a,t)$ and $\xi_m \in (a,c)$ for all m. $\square$

Iterated ABC differintegrals to arbitrary order would be very difficult to compute directly. Fortunately, we can use the series formula from Theorem 2.4.8 to derive a significantly simpler expression for $\left({}^{ABC}_{\ a}D^{\alpha} \right)^m f$, as follows:

$$\begin{aligned} \left({}^{ABC}_{\ a}D^{\alpha}_{t} \right)^m f(t) &= \left[\frac{B(\alpha)}{1-\alpha} \sum_{n=0}^{\infty} \left(\frac{-\alpha}{1-\alpha} \right)^n {}^{RL}_{\ a}I^{\alpha n+1}_{t} \frac{\mathrm{d}}{\mathrm{d}t} \right]^m f(t) \\ &= \frac{B(\alpha)^m}{(1-\alpha)^m} \sum_{n_1,\dots,n_m} \left(\frac{-\alpha}{1-\alpha} \right)^{\sum n_i} {}^{RL}_{\ a}I^{\alpha \sum n_i + 1}_{t} \frac{\mathrm{d}}{\mathrm{d}t} f(t) \\ &= \frac{B(\alpha)^m}{(1-\alpha)^m} \sum_{N=0}^{\infty} \binom{N+m-1}{m-1} \left(\frac{-\alpha}{1-\alpha} \right)^N {}^{RL}_{\ a}I^{\alpha N+1}_{t} f'(t), \end{aligned} \tag{111}$$

where this series is locally uniformly convergent in t. Using the formula (111) for the iterated ABC derivative makes the Taylor series (105) and (109) easier to compute for specific individual functions f.

Unfortunately, given the complexity of the formula for the remainder term R_{n+1}, it will be difficult to tell whether and when the series (109) converges as n goes to infinity. But we certainly have a valid finite series result, which can be verified computationally even for large values of n.

Example 2.4.21. As an illustrative example of Theorem 2.4.20, let us consider what the series looks like with the particular function $f(t) = (t-a)^{\beta}$.

Using the expression (111) for the iterated ABC derivative, we find that in this case

$$\left({}^{ABC}_{\ a}D^{\alpha}_{t} \right)^m f(t) \\ = \left(\frac{B(\alpha)}{1-\alpha} \right)^m \sum_{N=0}^{\infty} \binom{N+m-1}{m-1} \left(\frac{-\alpha}{1-\alpha} \right)^N \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha N+1)} (t-a)^{\beta+\alpha N} \tag{112}$$

So the ABC Taylor series for this $f(t)$ is given by (109) with the iterated ABC derivatives and the coefficients Δ_m given respectively by (112) and (110). I.e.:

$$f(t) = \sum_{m=0}^{n} \Delta_m \left(\tfrac{B(\alpha)}{1-\alpha}\right)^m \sum_{N=0}^{\infty} \tbinom{N+m-1}{m-1} \left(\frac{-\alpha}{1-\alpha}\right)^N \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha N+1)} (c-a)^{\beta+\alpha N}$$
$$+ \Bigg[\Delta_{0,n+1} \left({}^{ABC}_{\ a}D^{\alpha}_{\xi}\right)^{n+1} (\xi - a)^{\beta}$$
$$- \sum_{k=0}^{n-1} \Delta_{k,k} S_{\alpha,n-k+1}(c-a) \left({}^{ABC}_{\ a}D^{\alpha}_{\xi_k}\right)^{n+1} (\xi_k - a)^{\beta}\Bigg], \quad (113)$$

where the Δ and S functions are defined by (110) and (106), and the constants $\xi, \xi_1, \ldots, \xi_{n-1}$ are in the interval $(a, \max(c, t))$. $\square$

2.5 Further general models of fractional calculus

2.5.1 An iterated Atangana–Baleanu model

In the previous chapter, we obtained a closed-form expression (111) for the result of iterating AB differintegration arbitrarily many times. Inspired by the recent paper [80], we can use this iteration formula to introduce a **new model of fractional calculus**, with two-parameter indices, which arises from taking multiple iterations of AB integrals. As we shall see, some of the properties of the AB model extend to this new iterated model, and it also has some properties which the original AB model lacks, such as a semigroup property.

As already discussed in §2.4.5 above, the semigroup property is a natural thing to consider for any model of fractional calculus. It has even been proposed by some scientists [109] as a criterion for deciding whether or not an operator is a fractional derivative. Despite not subscribing to this more restrictive view of fractional calculus, I still feel it is worth considering this modification of the AB model which has a semigroup property.

The expression (58) for AB fractional integrals can be rewritten in distributional form as follows:

$$\begin{aligned}{}^{AB}_{\ a}I^{\alpha}_{t}f(t)&=\frac{1-\alpha}{B(\alpha)}f(t)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^t(t-x)^{\alpha-1}f(x)\,\mathrm{d}x\\&=\int_a^t f(x)\left[\frac{1-\alpha}{B(\alpha)}\delta(t-x)+\frac{\alpha}{B(\alpha)\Gamma(\alpha)}(t-x)^{\alpha-1}\right]\mathrm{d}x,\end{aligned}$$

where δ is the Dirac delta function.

Iterating the AB integral an arbitrary natural number of times gives the following formula for sequential AB fractional integrals:

$$\begin{aligned}\left({}^{AB}_{\ a}I^{\alpha}_{t}\right)^n f(t)&=\left[\frac{1-\alpha}{B(\alpha)}+\frac{\alpha}{B(\alpha)}\,{}^{RL}_{\ a}I^{\alpha}_{t}\right]^n f(t)\\&=\sum_{k=0}^{n}\frac{\binom{n}{k}(1-\alpha)^{n-k}\alpha^k}{B(\alpha)^n}\,{}^{RL}_{\ a}I^{\alpha k}_{t}f(t) &(114)\\&=\left(\frac{1-\alpha}{B(\alpha)}\right)^n f(t)+\sum_{k=1}^{n}\frac{\binom{n}{k}(1-\alpha)^{n-k}\alpha^k}{B(\alpha)^n\Gamma(k\alpha)}\int_a^t(t-x)^{k\alpha-1}f(x)\,\mathrm{d}x, &(115)\end{aligned}$$

where we have used the semigroup property for Riemann–Liouville integrals, Lemma 1.1.6. This formula too can be written in distributional form, as follows:

$$\left({}^{AB}_{\ a}I^{\alpha}_{t}\right)^n f(t)=\int_a^t f(x)\left[\left(\frac{1-\alpha}{B(\alpha)}\right)^n\delta(t-x)+\sum_{k=1}^{n}\frac{\binom{n}{k}(1-\alpha)^{n-k}\alpha^k}{B(\alpha)^n\Gamma(k\alpha)}(t-x)^{k\alpha-1}\right]\mathrm{d}x. \tag{116}$$

The series in equations (114)-(116) is a finite binomial series arising from the nth power. Thus it is easy to generalise to arbitrary powers, using an infinite binomial series. We define the βth iteration of the αth AB integral, for $0 < \alpha < 1$ and $\beta \in \mathbb{R}$, by the following equivalent formulae. (We include all three of these formulae, because they are clearly equivalent, and each one of them can be more useful than the others according to the particular context in question.)

$$\left({}^{AB}_{\ a}I^{\alpha}_{t}\right)^{\beta} f(t) = \sum_{k=0}^{\infty} \frac{\binom{\beta}{k}(1-\alpha)^{\beta-k}\alpha^k}{B(\alpha)^{\beta}} \, {}^{RL}_{\ a}I^{\alpha k}_{t} f(t) \tag{117}$$

$$= \left(\frac{1-\alpha}{B(\alpha)}\right)^{\beta} f(t) + \sum_{k=1}^{\infty} \frac{\binom{\beta}{k}(1-\alpha)^{\beta-k}\alpha^k}{B(\alpha)^{\beta}\Gamma(k\alpha)} \int_a^t (t-x)^{k\alpha-1} f(x) \,\mathrm{d}x \tag{118}$$

$$= \int_a^t \left(\frac{1-\alpha}{B(\alpha)}\right)^{\beta} f(x) \left[\delta(t-x) + \sum_{k=1}^{\infty} \frac{\binom{\beta}{k}(1-\alpha)^{-k}\alpha^k}{\Gamma(k\alpha)}(t-x)^{k\alpha-1}\right] \mathrm{d}x. \tag{119}$$

Note that these expressions exist regardless of the sign of β: our definition is a true fractional differintegral, treating derivatives and integrals equally. We formalise the definition as follows.

Definition 2.5.1. Let $0 \leq \alpha \leq 1$, $\beta \in \mathbb{R}$, $a < b$ in $\mathbb{R}$, and $f : [a, b] \to \mathbb{R}$ be an L^1 function. The βth iteration of the αth AB integral of a function f, which we shall call an **iterated AB differintegral** and denote by $\mathcal{I}^{(\alpha,\beta)}_{a+} f(t)$, is defined by the formulae (117)-(119). In other words, the iterated AB integral is given by

$$\mathcal{I}^{(\alpha,\beta)}_{a+} f(t) = \sum_{k=0}^{\infty} \frac{\binom{\beta}{k}(1-\alpha)^{\beta-k}\alpha^k}{B(\alpha)^{\beta}} \, {}^{RL}_{\ a}I^{\alpha k}_{t} f(t), \tag{120}$$

and the iterated AB derivative is given by

$$\mathcal{D}^{(\alpha,\beta)}_{a+} f(t) = \sum_{k=0}^{\infty} \frac{\binom{-\beta}{k}\alpha^k B(\alpha)^{\beta}}{(1-\alpha)^{\beta+k}} \, {}^{RL}_{\ a}I^{\alpha k}_{t} f(t). \tag{121}$$

Remark 2.5.2. In order to demonstrate the appropriateness of Definition 2.5.1, we consider how this differintegral behaves in different specific cases of the variables α and β.

- If $\alpha = 0$, then the operator is trivial:

$$\mathcal{I}^{(0,\beta)}_{a+} f(t) = f(t).$$

- If $\beta = 0$, then the operator is trivial:

$$\mathcal{I}^{(\alpha,0)}_{a+} f(t) = f(t).$$

- If $\beta = n \in \mathbb{N}$, then the original formulae (114)-(116) for iterated AB integrals are recovered:
$$\mathcal{I}_{a+}^{(\alpha,n)} f(t) = \left({}^{AB}_{\ a}I^{\alpha}_{t}\right)^{n} f(t).$$

- If $\beta = -1$, then the operator is the ABR derivative, because (117) becomes the series expression (65), while (119) is analogous to the distributional formulation of the ABC derivative used in §2.4.7 above:
$$\mathcal{I}_{a+}^{(\alpha,-1)} f(t) = {}^{ABR}_{\ \ a}D^{\alpha}_{t} f(t).$$

- If $\beta = -n$, $n \in \mathbb{N}$, then similarly the operator is the iterated ABR derivative:
$$\mathcal{I}_{a+}^{(\alpha,-n)} f(t) = \left({}^{ABR}_{\ \ a}D^{\alpha}_{t}\right)^{n} f(t).$$

We shall now prove some basic properties of our new definition. In particular, we note that convergence of the series (120) and (121) is given by the boundedness of the associated operators, proved in Theorem 2.5.5 below.

Theorem 2.5.3 (Laplace transforms)**.** *If α, β, a, b, and f are as in Definition 2.5.1 and f has a well-defined Laplace transform, then the Laplace transform of its iterated AB differintegral is given by*
$$\mathcal{L}\left(\mathcal{I}_{0+}^{(\alpha,\beta)} f(t)\right) = \left(\frac{1-\alpha}{B(\alpha)} + \frac{\alpha}{B(\alpha)} s^{-\alpha}\right)^{\beta} \hat{f}(s), \tag{122}$$
where $\mathcal{L}$ and $\hat{}$ both denote the Laplace transform.

Proof. This follows from the formula (117), since we know what the Laplace transforms of Riemann–Liouville fractional operators look like:
$$\begin{aligned}
\mathcal{L}\left(\mathcal{I}_{0+}^{(\alpha,\beta)} f(t)\right) &= \mathcal{L}\left(\sum_{k=0}^{\infty} \frac{\binom{\beta}{k}(1-\alpha)^{\beta-k}\alpha^{k}}{B(\alpha)^{\beta}}\, {}^{RL}_{\ \ 0}I^{\alpha k}_{t} f(t)\right) \\
&= \sum_{k=0}^{\infty} \frac{\binom{\beta}{k}(1-\alpha)^{\beta-k}\alpha^{k}}{B(\alpha)^{\beta}} \mathcal{L}\left({}^{RL}_{\ \ 0}I^{\alpha k}_{t} f(t)\right) \\
&= \sum_{k=0}^{\infty} \frac{\binom{\beta}{k}(1-\alpha)^{\beta-k}\alpha^{k}}{B(\alpha)^{\beta}} s^{-\alpha k} \hat{f}(s) \\
&= \sum_{k=0}^{\infty} \frac{\binom{\beta}{k}(1-\alpha)^{\beta-k}(\alpha s^{-\alpha})^{k}}{B(\alpha)^{\beta}} \hat{f}(s) \\
&= \left(\frac{1-\alpha}{B(\alpha)} + \frac{\alpha}{B(\alpha)} s^{-\alpha}\right)^{\beta} \hat{f}(s),
\end{aligned}$$
where for the last step we used the binomial theorem again. □

A very important aspect to consider for any fractional differintegral is the semigroup property, i.e. the question of whether or not a differintegral of a differintegral is a differintegral of the expected order. We know from the classical results Lemmas 1.1.6–1.1.7 that in the Riemann–Liouville model, fractional integrals satisfy the semigroup property but fractional derivatives in general do not. We also know from §2.4.5 above that in the AB model, neither derivatives nor integrals satisfy the semigroup property. By contrast, in the new iterated AB model, there is a semigroup property in β for all differintegrals.

Theorem 2.5.4 (Semigroup property)**.** *Iterated AB differintegrals have a semigroup property in β, i.e.*

$$\mathcal{I}_{a+}^{(\alpha,\beta)}\mathcal{I}_{a+}^{(\alpha,\gamma)}f(t)=\mathcal{I}_{a+}^{(\alpha,\beta+\gamma)}f(t) \tag{123}$$

for all $\alpha\in[0,1]$, $\beta,\gamma\in\mathbb{R}$, and a,f as in Definition 2.5.1.

Proof. Once again, this is a consequence of the fact that our new model is derived from binomial expansions. We use the formula (117) and the fact that Riemann–Liouville integrals have a semigroup property:

$$\begin{aligned}
\mathcal{I}_{a+}^{(\alpha,\beta)}\mathcal{I}_{a+}^{(\alpha,\gamma)}f(t)&=\sum_{k=0}^{\infty}\frac{\binom{\beta}{k}(1-\alpha)^{\beta-k}\alpha^k}{B(\alpha)^{\beta}}\,{}_{a}^{RL}I_t^{\alpha k}\left[\sum_{j=0}^{\infty}\frac{\binom{\gamma}{j}(1-\alpha)^{\gamma-j}\alpha^j}{B(\alpha)^{\gamma}}\,{}_{a}^{RL}I_t^{\alpha j}f(t)\right]\\
&=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{\binom{\beta}{k}\binom{\gamma}{j}(1-\alpha)^{(\beta+\gamma)-(k+j)}\alpha^{k+j}}{B(\alpha)^{\beta+\gamma}}\,{}_{a}^{RL}I_t^{\alpha(k+j)}f(t)\\
&=\sum_{m=0}^{\infty}\sum_{k=0}^{m}\frac{\binom{\beta}{k}\binom{\gamma}{m-k}(1-\alpha)^{(\beta+\gamma)-m}\alpha^{m}}{B(\alpha)^{\beta+\gamma}}\,{}_{a}^{RL}I_t^{\alpha m}f(t)\\
&=\sum_{m=0}^{\infty}\frac{\binom{\beta+\gamma}{m}(1-\alpha)^{(\beta+\gamma)-m}\alpha^{m}}{B(\alpha)^{\beta+\gamma}}\,{}_{a}^{RL}I_t^{\alpha m}f(t)=\mathcal{I}_{a+}^{(\alpha,\beta+\gamma)}f(t),
\end{aligned}$$

where we have used the binomial identity $\sum_{k=0}^{m}\binom{\beta}{k}\binom{\gamma}{m-k}=\binom{\beta+\gamma}{m}$. □

We also show that all differintegral operators in the new model are bounded in the L^1 and L^∞ norms.

Theorem 2.5.5 (Bounded operators)**.** *Let a,b,α,β be as in Definition 2.5.1. There exists a positive constant K such that for any $f\in L^1[a,b]$,*

$$||\mathcal{I}_{a+}^{(\alpha,\beta)}f||_1\leq K||f||_1, \tag{124}$$

and, if we also assume f is continuous,

$$||\mathcal{I}_{a+}^{(\alpha,\beta)}f||_\infty\leq K||f||_\infty. \tag{125}$$

Proof. We use the formula (118) to find bounds on $\mathcal{I}_{a+}^{(\alpha,\beta)}f(t)$.

By the first mean value theorem for integrals, provided f is continuous (and therefore bounded), we have

$$\int_a^t (t-x)^{k\alpha-1} f(x)\,\mathrm{d}x = f(c)\int_a^t (t-x)^{k\alpha-1}\,\mathrm{d}x = f(c)\frac{(t-a)^{k\alpha}}{k\alpha}$$

for some $c \in (a,t)$, and therefore

$$\left|\int_a^t (t-x)^{k\alpha-1} f(x)\,\mathrm{d}x\right| \le ||f||_\infty \frac{(b-a)^{k\alpha}}{k\alpha}.$$

Thus, the formula (118) gives

$$\begin{aligned}
\left|\mathcal{I}_{a+}^{(\alpha,\beta)} f(t)\right| &= \left|\left(\frac{1-\alpha}{B(\alpha)}\right)^\beta f(t) + \sum_{k=1}^\infty \frac{\binom{\beta}{k}(1-\alpha)^{\beta-k}\alpha^k}{B(\alpha)^\beta\Gamma(k\alpha)}\int_a^t (t-x)^{k\alpha-1} f(x)\,\mathrm{d}x\right| \\
&\le \left(\frac{1-\alpha}{B(\alpha)}\right)^\beta ||f||_\infty + \sum_{k=1}^\infty \frac{\left|\binom{\beta}{k}\right|(1-\alpha)^{\beta-k}\alpha^k}{B(\alpha)^\beta\Gamma(k\alpha)}\left|\int_a^t (t-x)^{k\alpha-1} f(x)\,\mathrm{d}x\right| \\
&\le \left[\left(\frac{1-\alpha}{B(\alpha)}\right)^\beta + \sum_{k=1}^\infty \frac{\left|\binom{\beta}{k}\right|(1-\alpha)^{\beta-k}\alpha^k(b-a)^{k\alpha}}{B(\alpha)^\beta\Gamma(k\alpha+1)}\right]||f||_\infty \\
&= \left[\left(\frac{1-\alpha}{B(\alpha)}\right)^\beta \sum_{k=0}^\infty \left|\binom{\beta}{k}\right|\left(\frac{\alpha(b-a)^\alpha}{1-\alpha}\right)^k \frac{1}{\Gamma(k\alpha+1)}\right]||f||_\infty.
\end{aligned}$$

The term in square brackets depends only on a, b, α, and β, so we have proved (125).

By the second mean value theorem for integrals, we have

$$\int_a^t (t-x)^{k\alpha-1}|f(x)|\,\mathrm{d}x = (t-a)^{k\alpha-1}\int_a^c |f(x)|\,\mathrm{d}x$$

for some $c \in (a,t]$, and therefore

$$\left|\int_a^t (t-x)^{k\alpha-1} f(x)\,\mathrm{d}x\right| \le ||f||_1 (t-a)^{k\alpha-1}.$$

Thus, the formula (118) gives

$$\left|\mathcal{I}_{a+}^{(\alpha,\beta)} f(t)\right| \le \left(\frac{1-\alpha}{B(\alpha)}\right)^\beta |f(t)| + \sum_{k=1}^\infty \frac{\left|\binom{\beta}{k}\right|(1-\alpha)^{\beta-k}\alpha^k}{B(\alpha)^\beta\Gamma(k\alpha)}||f||_1 (t-a)^{k\alpha-1}.$$

Integrating this inequality with respect to t yields

$$\int_a^b \left|\mathcal{I}_{a+}^{(\alpha,\beta)} f(t)\right|\,\mathrm{d}t \le \left(\frac{1-\alpha}{B(\alpha)}\right)^\beta \int_a^b |f(t)|\,\mathrm{d}t + \sum_{k=1}^\infty \frac{\left|\binom{\beta}{k}\right|(1-\alpha)^{\beta-k}\alpha^k}{B(\alpha)^\beta\Gamma(k\alpha)}||f||_1 \frac{(b-a)^{k\alpha}}{k\alpha},$$

and therefore $||\mathcal{I}_{a+}^{(\alpha,\beta)} f||_1 \le K||f||_1$ with the constant K being exactly the same as before.

$\square$

Finally, we consider certain classes of fractional ordinary differintegral equations which can be solved in the new model. For example, let us solve the following equation, of a similar (but more general) form to one considered in the AB model in §2.4.4 above:

$$\mathcal{I}_{0+}^{(\alpha,\beta)}f(t) = P + Qf(t) + R(f(t))^2, \tag{126}$$

where α, β, P, Q, R are fixed. We use a series solution method with the following ansatz:

$$f(t) = \sum_{n=0}^{\infty} a_n t^{n\alpha}. \tag{127}$$

When $f(t)$ is in this form, we use the formula (117) to evaluate $\mathcal{I}_{0+}^{(\alpha,\beta)}f(t)$:

$$\begin{aligned}
\mathcal{I}_{0+}^{(\alpha,\beta)}f(t) &= \sum_{k=0}^{\infty} \frac{\binom{\beta}{k}(1-\alpha)^{\beta-k}\alpha^k}{B(\alpha)^\beta} \, {}^{RL}_{\ 0}I_t^{\alpha k}\left(\sum_{l=0}^{\infty} a_l t^{l\alpha}\right) \\
&= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{\binom{\beta}{k}(1-\alpha)^{\beta-k}\alpha^k}{B(\alpha)^\beta} a_l \frac{\Gamma(l\alpha+1)}{\Gamma((k+l)\alpha+1)} t^{(k+l)\alpha} \\
&= \sum_{m=0}^{\infty} \frac{t^{m\alpha}}{B(\alpha)^\beta \Gamma(m\alpha+1)} \sum_{k=0}^{m} a_{m-k}\binom{\beta}{k}(1-\alpha)^{\beta-k}\alpha^k \Gamma((m-k)\alpha+1).
\end{aligned} \tag{128}$$

This is the left-hand side of the equation (126), while the right-hand side is:

$$\begin{aligned}
P + Qf(t) + R(f(t))^2 &= P + Q\sum_{m=0}^{\infty} a_m t^{m\alpha} + R\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a_k a_l t^{(k+l)\alpha} \\
&= \sum_{m=0}^{\infty}\left(P\delta_{m0} + Qa_m + R\sum_{k=0}^{m} a_k a_{m-k}\right) t^{m\alpha}.
\end{aligned} \tag{129}$$

Equating coefficients in (128) and (129), we find for $m = 0$ that

$$a_0 \left(\frac{1-\alpha}{B(\alpha)}\right)^\beta = P + Qa_0 + Ra_0^2$$

and therefore

$$a_0 = \frac{\left(\frac{1-\alpha}{B(\alpha)}\right)^\beta - Q \pm \sqrt{\left[\left(\frac{1-\alpha}{B(\alpha)}\right)^\beta - Q\right]^2 - 4PR}}{2R}, \tag{130}$$

while for $m > 0$ we have

$$\frac{1}{\Gamma(m\alpha+1)B(\alpha)^\beta}\Big[a_m(1-\alpha)^\beta\Gamma(m\alpha+1) \\ +\sum_{k=1}^{m} a_{m-k}\tbinom{\beta}{k}(1-\alpha)^{\beta-k}\alpha^k\Gamma((m-k)\alpha+1)\Big] = Qa_m + R\left[2a_ma_0+\sum_{k=1}^{m-1}a_ka_{m-k}\right]$$

and therefore

$$a_m\left[\left(\frac{1-\alpha}{B(\alpha)}\right)^\beta - Q - 2Ra_0\right] \\ = R\sum_{k=1}^{m-1}a_ka_{m-k} - \frac{\sum_{k=1}^{m} a_{m-k}\binom{\beta}{k}(1-\alpha)^{\beta-k}\alpha^k\Gamma((m-k)\alpha+1)}{\Gamma(m\alpha+1)B(\alpha)^\beta}.$$

And the formula (130) for a_0 enables us to simplify the a_m coefficient here. Thus, we derive the following general expression for the solution $f(t)$ of (126):

$$f(t) = a_0 + \sum_{m=1}^{\infty}\left[\frac{R\sum_{k=1}^{m-1}a_ka_{m-k} - \frac{\sum_{k=1}^{m} a_{m-k}\binom{\beta}{k}(1-\alpha)^{\beta-k}\alpha^k\Gamma((m-k)\alpha+1)}{\Gamma(m\alpha+1)B(\alpha)^\beta}}{\mp\left(\left[\left(\frac{1-\alpha}{B(\alpha)}\right)^\beta - Q\right]^2 - 4PR\right)^{1/2}}\right]t^{m\alpha}, \tag{131}$$

where the constant term a_0 is given by (130).

2.5.2 A series formula for the Prabhakar model

An operator introduced by Prabhakar in 1971 [122] for solving a particular singular integral equation has also been adapted as a fractional differintegral operator [86], and its properties and applications have been investigated in papers such as [86, 61, 67]. Although it predates the Atangana–Baleanu model, it can be seen as an extension thereof, and some of our analysis of the AB model from §2.4 can be applied equally well to the Prabhakar model.

Definition 2.5.6. The **Prabhakar** fractional integral is defined by

$$\mathcal{E}^{\omega,\rho}_{\alpha,\beta;c+}f(x) \coloneqq \int_c^x (x-t)^{\beta-1}E^\rho_{\alpha,\beta}\left[\omega(x-t)^\alpha\right]f(t)\,\mathrm{d}t, \quad \mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0, \tag{132}$$

where the generalised Mittag-Leffler function $E^\rho_{\alpha,\beta}$ is defined by

$$E^\rho_{\alpha,\beta}(z) \coloneqq \sum_{n=0}^{\infty}\frac{(\rho)_nz^n}{\Gamma(\alpha n+\beta)n!} = \sum_{n=0}^{\infty}\frac{\Gamma(\rho+n)z^n}{\Gamma(\rho)\Gamma(\alpha n+\beta)n!}. \tag{133}$$

The Prabhakar integral is known [86] to be a bounded operator on L^1 functions. It has also been shown [86] that its left inverse can be given by any expression of the following form:

$$\left[\mathcal{E}^{\omega,\rho}_{\alpha,\beta;c+}\right]^{-1} f(x) = {}^{RL}_{\ c}D^{\beta+\gamma}_{x}\mathcal{E}^{\omega,-\rho}_{\alpha,\gamma;c+}f(x), \quad \gamma\in\mathbb{C}, \mathrm{Re}(\gamma)>0. \tag{134}$$

In particular, and by analogy with the definition (2) of Riemann–Liouville fractional derivatives, the Prabhakar fractional differential operator can be defined as follows:

$$\mathcal{D}^{\omega,\rho}_{\alpha,\beta;c+}f(x) \coloneqq \frac{\mathrm{d}^m}{\mathrm{d}x^m}\mathcal{E}^{\omega,-\rho}_{\alpha,m-\beta;c+}f(x), \quad \mathrm{Re}(\alpha)>0, \mathrm{Re}(\beta)>0, m\coloneqq\lfloor\mathrm{Re}(\beta)\rfloor+1. \tag{135}$$

The Prabhakar operator has also been extended and generalised still further [133, 62]. The **generalised Prabhakar** fractional integral is defined by

$$\mathcal{E}^{\omega,\rho,\kappa}_{\alpha,\beta;c+}f(x) \coloneqq \int_c^x (x-t)^{\beta-1}E^{\rho,\kappa}_{\alpha,\beta}\left[\omega(x-t)^{\alpha}\right]f(t)\,\mathrm{d}t,$$
$$\min\left(\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\kappa)\right)>0, \mathrm{Re}(\kappa-\alpha)<1, \tag{136}$$

where the generalised Mittag-Leffler function $E^{\rho,\kappa}_{\alpha,\beta}$ is defined by

$$E^{\rho,\kappa}_{\alpha,\beta}(z) \coloneqq \sum_{n=0}^{\infty}\frac{(\rho)_{\kappa n}z^n}{\Gamma(\alpha n+\beta)n!} = \sum_{n=0}^{\infty}\frac{\Gamma(\rho+\kappa n)z^n}{\Gamma(\rho)\Gamma(\alpha n+\beta)n!}. \tag{137}$$

The generalised Prabhakar integral is known [133] to be a bounded operator on L^1 functions.

The methodology we used in §2.4 to prove a series formula for AB derivatives can be adapted to prove analogous results for the Prabhakar and generalised Prabhakar models. Furthermore, once again the series formulae in these models can be used to give quick alternative proofs for several known results on Prabhakar differintegrals, and also to derive new results in the Prabhakar model such as analogues of the product rule and chain rule. All of these results in the Prabhakar model lead directly to similar results for other types of fractional calculus such as the CF, AB, and iterated AB models, which can all be seen as special cases of Prabhakar.

Our starting point is the series formula (137) for the generalised Mittag-Leffler function. This series is known [86, 91] to be locally uniformly convergent in z, provided that

$$\mathrm{Re}(\alpha)>0, \quad \mathrm{Re}(\beta)>0, \quad \mathrm{Re}(\kappa)>0, \quad \mathrm{Re}(\kappa-\alpha)<1. \tag{138}$$

Thus we can interchange the summation and integration in the formula (136), to get:

$$\begin{aligned}
\mathcal{E}^{\omega,\rho,\kappa}_{\alpha,\beta;c+}f(x) &= \int_c^x (x-t)^{\beta-1}E^{\rho,\kappa}_{\alpha,\beta}\left[\omega(x-t)^{\alpha}\right]f(t)\,\mathrm{d}t \\
&= \int_c^x (x-t)^{\beta-1}\sum_{n=0}^{\infty}\frac{(\rho)_{\kappa n}\omega^n(x-t)^{\alpha n}}{\Gamma(\alpha n+\beta)n!}f(t)\,\mathrm{d}t \\
&= \sum_{n=0}^{\infty}\int_c^x \frac{\Gamma(\rho+\kappa n)\omega^n(x-t)^{\alpha n+\beta-1}}{\Gamma(\rho)\Gamma(\alpha n+\beta)n!}f(t)\,\mathrm{d}t \\
&= \sum_{n=0}^{\infty}\frac{\Gamma(\rho+\kappa n)\omega^n}{\Gamma(\rho)n!}\cdot\frac{1}{\Gamma(\alpha n+\beta)}\int_c^x (x-t)^{\alpha n+\beta-1}f(t)\,\mathrm{d}t \\
&= \sum_{n=0}^{\infty}\frac{\Gamma(\rho+\kappa n)\omega^n}{\Gamma(\rho)n!}\,{}^{RL}_{\ c}I^{\alpha n+\beta}_{x}f(x).
\end{aligned}$$

Now we have expressed the generalised Prabhakar operator as a series of Riemann–Liouville fractional integrals, and the following result is established.

Theorem 2.5.7. *Under the conditions* (138) *on the parameters* $\alpha,\beta,\omega,\rho,\kappa$, *for any interval* $(c,d)\subset\mathbb{R}$ *and any function* $f\in L^1(c,d)$, *the generalised Prabhakar operator can be written as*

$$\mathcal{E}^{\omega,\rho,\kappa}_{\alpha,\beta;c+}f(x)=\sum_{n=0}^{\infty}\frac{\Gamma(\rho+\kappa n)\omega^n}{\Gamma(\rho)n!}\,{}^{RL}_{\ c}I^{\alpha n+\beta}_{x}f(x), \tag{139}$$

where the series on the right-hand side is locally uniformly convergent.

In order to deduce similar results in other models of fractional calculus, we first note the following equivalences.

Proposition 2.5.8. *The Prabhakar, CF, AB, and iterated AB models of fractional calculus can all be viewed as special cases of the generalised Prabhakar model* (136), *as follows.*

$$\mathcal{E}^{\omega,\rho}_{\alpha,\beta;c+}f(x)=\mathcal{E}^{\omega,\rho,1}_{\alpha,\beta;c+}f(x), \qquad \mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0; \tag{140}$$

$${}^{CF}_{\ c}D^{\alpha}_{x}f(x)=\frac{M(\alpha)}{1-\alpha}\mathcal{E}^{\frac{-\alpha}{1-\alpha},1,1}_{1,1;c+}f'(x), \qquad 0<\alpha<1; \tag{141}$$

$${}^{ABR}_{\ \ c}D^{\alpha}_{x}f(x)=\frac{B(\alpha)}{1-\alpha}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\mathcal{E}^{\frac{-\alpha}{1-\alpha},1,1}_{\alpha,1;c+}f(x), \qquad 0<\alpha<1; \tag{142}$$

$${}^{ABC}_{\ \ c}D^{\alpha}_{x}f(x)=\frac{B(\alpha)}{1-\alpha}\mathcal{E}^{\frac{-\alpha}{1-\alpha},1,1}_{\alpha,1;c+}f'(x), \qquad 0<\alpha<1; \tag{143}$$

$$\mathcal{I}^{\alpha,\rho}_{c+}f(x)=\left(\frac{1-\alpha}{B(\alpha)}\right)^{\rho}\mathcal{E}^{\frac{\alpha}{1-\alpha},\rho+1,1}_{\alpha,0;c+}f(x), \qquad 0\le\alpha\le 1,\rho\in\mathbb{R}. \tag{144}$$

Proof. This follows directly from comparing (132) and Definitions 1.1.14, 2.4.1, 2.4.2, 2.5.1 with the formula (136) for the generalised Prabhakar integral. □

Using Proposition 2.5.8, it is straightforward to deduce the following corollaries of Theorem 2.5.7 for the other models of fractional calculus which can be seen as special cases of generalised Prabhakar.

Corollary 2.5.9. *Given complex parameters* $\alpha, \beta, \omega, \rho$ *satisfying* $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0$, *then for any interval* $(c, d) \subset \mathbb{R}$ *and any function* $f \in L^1(c, d)$*, the Prabhakar fractional integral can be written as*

$$\mathcal{E}^{\omega,\rho}_{\alpha,\beta;c+} f(x) = \sum_{n=0}^{\infty} \frac{\Gamma(\rho + n)\omega^n}{\Gamma(\rho)n!} \, {}^{RL}_{\ c}I^{\alpha n+\beta}_{x} f(x), \tag{145}$$

and the Prabhakar fractional derivative can be written as

$$\mathcal{D}^{\omega,\rho}_{\alpha,\beta;c+} f(x) = \sum_{n=0}^{\infty} \frac{\Gamma(-\rho + n)\omega^n}{\Gamma(-\rho)n!} \, {}^{RL}_{\ c}I^{\alpha n-\beta}_{x} f(x), \tag{146}$$

where the series on the right-hand sides are locally uniformly convergent.

We note that if (145)-(146) are used as the definitions of Prabhakar fractional integrals and derivatives, then differintegrals in this model can be unified under a single series formula, where switching between derivatives to integrals means simply switching the sign of the parameters ρ and β.

Corollary 2.5.10. *Given* $\alpha \in (0, 1)$*, then for any interval* $(c, d) \subset \mathbb{R}$ *and any function* $f \in L^1(c, d)$*, the Caputo–Fabrizio fractional derivative can be written as*

$${}^{CF}_{\ c}D^{\alpha}_{x} f(x) = \frac{M(\alpha)}{1 - \alpha} \sum_{n=0}^{\infty} \left(\frac{-\alpha}{1 - \alpha} \right)^n {}^{RL}_{\ c}I^{n+1}_{x} f'(x), \tag{147}$$

where the series on the right-hand side is locally uniformly convergent.

Remark 2.5.11. We note that the series expression (147) does not even involve fractional integrals: the CF fractional derivative can be written simply as a convergent series of classical iterated integrals.

Our previous results on series formulae in the AB model, Theorems 2.4.4 and 2.4.8, can also be derived as corollaries of Theorem 2.5.7. For the iterated AB differintegral, we have no new result, since substituting (144) into (139) would simply yield the series (120) which was used to define the operator in the first place.

As with the AB model, we also expect that the series formula will make numerical computation of Prabhakar derivatives easier than before. For such computation we now only need to consider series of RL fractional integrals, not special functions such as the generalised Mittag-Leffler function.

The series formulae we have now established make the proofs much easier for several fundamental results about Prabhakar operators and their relationships with classical fractional operators. For example, the following result comprises Theorems 3 and 4 in [122], Theorems 6 and 7 in [86], and Theorems 4 and 5 in [133]. Each of these identities was proved originally using Fubini's theorem, but now follows much more quickly from the new series formula.

Theorem 2.5.12. *The generalised Prabhakar operator* (136)*, with* $\alpha,\beta,\omega,\rho,\kappa\in\mathbb{C}$ *satisfying* (138)*, interacts naturally with Riemann–Liouville differintegral operators in the following ways.*

For any L^1 *function* f*, and any* $\mu\in\mathbb{C}$ *such that* $\mathrm{Re}(\mu)>-\mathrm{Re}(\beta)$*, we have:*

$$ {}^{RL}_{c}I^{\mu}\mathcal{E}^{\omega,\rho,\kappa}_{\alpha,\beta;c+}f=\mathcal{E}^{\omega,\rho,\kappa}_{\alpha,\beta+\mu;c+}f. \tag{148}$$

If in addition $\mathrm{Re}(\mu)>0$*, then*

$$ {}^{RL}_{c}I^{\mu}\mathcal{E}^{\omega,\rho,\kappa}_{\alpha,\beta;c+}f=\mathcal{E}^{\omega,\rho,\kappa}_{\alpha,\beta;c+}\,{}^{RL}_{c}I^{\mu}f. \tag{149}$$

Proof. For the first identity, by the series formula (139) it is enough to show that

$$ {}^{RL}_{c}I^{\mu}\,{}^{RL}_{c}I^{\alpha n+\beta}f={}^{RL}_{c}I^{\alpha n+\beta+\mu}f,\quad n\geq 0,$$

which is clearly true by the standard Lemma 1.1.6. For the second identity, by (139) it is enough to show that

$$ {}^{RL}_{c}I^{\mu}\,{}^{RL}_{c}I^{\alpha n+\beta}f={}^{RL}_{c}I^{\alpha n+\beta}\,{}^{RL}_{c}I^{\mu}f,\quad n\geq 0,$$

which again is just a case of Lemma 1.1.6 on the RL semigroup property. □

Remark 2.5.13. It is important to note that (149) is *not* always valid when $\mathrm{Re}(\mu)<0$. It will be valid under certain initial value conditions on f, but in general the left and right hand sides differ by a series of initial value terms, just as in the Riemann–Liouville case (Lemma 1.1.7).

The following result comprises Theorem 5 in [122] and Theorem 8 in [86], both of which were originally proved using Fubini's theorem. It can now be proved in a more elementary way using our Theorem 2.5.7.

Theorem 2.5.14. *The Prabhakar operator* (132) *satisfies the following semigroup property, under the usual assumptions* $\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta_1)>0,\mathrm{Re}(\beta_2)>0$ *and operating on an* L^1 *function* f*.*

$$\mathcal{E}^{\omega,\rho_1}_{\alpha,\beta_1;c+}\mathcal{E}^{\omega,\rho_2}_{\alpha,\beta_2;c+}f=\mathcal{E}^{\omega,\rho_1+\rho_2}_{\alpha,\beta_1+\beta_2;c+}f. \tag{150}$$

Proof. Using the series formula (145), the left-hand side of (150) becomes

$$\begin{aligned}
\mathcal{E}^{\omega,\rho_1}_{\alpha,\beta_1;c+}\mathcal{E}^{\omega,\rho_2}_{\alpha,\beta_2;c+}f(x) &= \sum_{n=0}^{\infty}\frac{\Gamma(\rho_1+n)\omega^n}{\Gamma(\rho_1)n!}\,{}^{RL}_{c}I^{\alpha n+\beta_1}_{x}\left(\sum_{m=0}^{\infty}\frac{\Gamma(\rho_2+m)\omega^m}{\Gamma(\rho_2)m!}\,{}^{RL}_{c}I^{\alpha m+\beta_2}_{x}f(x)\right) \\
&= \sum_{m,n}\frac{\Gamma(\rho_1+n)\Gamma(\rho_2+m)\omega^{m+n}}{\Gamma(\rho_1)\Gamma(\rho_2)n!m!}\,{}^{RL}_{c}I^{\alpha(m+n)+\beta_1+\beta_2}_{x} \\
&= \sum_{k=0}^{\infty}\left[\sum_{m+n=k}\frac{B(\rho_1+n,\rho_2+m)(m+n)!}{B(\rho_1,\rho_2)n!m!}\right] \\
&\qquad\qquad \frac{\Gamma(\rho_1+\rho_2+k)\omega^k}{\Gamma(\rho_1+\rho_2)k!}\,{}^{RL}_{c}I^{\alpha k+\beta_1+\beta_2}_{x},
\end{aligned}$$

where B is the beta function. Similarly, the right-hand side becomes

$$\mathcal{E}^{\omega,\rho_1+\rho_2}_{\alpha,\beta_1+\beta_2;c+}f(x) = \sum_{k=0}^{\infty}\frac{\Gamma(\rho_1+\rho_2+k)\omega^k}{\Gamma(\rho_1+\rho_2)k!}\,{}^{RL}_{c}I^{\alpha k+\beta_1+\beta_2}_{x}f(x).$$

So it suffices to prove that

$$\sum_{m+n=k}\frac{B(\rho_1+n,\rho_2+m)(m+n)!}{B(\rho_1,\rho_2)n!m!} = 1,$$

which can be verified by induction on k, using the fact that

$$B(x,y) = \frac{x-1}{x+y-1}B(x-1,y) + \frac{y-1}{x+y-1}B(x,y-1).$$

□

The following result comprises Theorem 9 in [86] and our equation (134), and it is used to justify the definition (135) of Prabhakar fractional derivatives.

Theorem 2.5.15. *Under the usual assumptions* $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0$*, the Prabhakar fractional integral defined by* (132) *has a left inverse on the space of* L^1 *functions. This left inverse can be defined by the expression* (134)*, which is independent of the value of* γ *and therefore equivalent to the Prabhakar fractional derivative* (135)*.*

Proof. Using first the result of Theorem 2.5.14 and then the series formula (145), we find that

$$\begin{aligned}
{}^{RL}_{c}D^{\beta+\gamma}_{x}\mathcal{E}^{\omega,-\rho}_{\alpha,\gamma;c+}\mathcal{E}^{\omega,\rho}_{\alpha,\beta;c+}f(x) &= {}^{RL}_{c}D^{\beta+\gamma}_{x}\mathcal{E}^{\omega,0}_{\alpha,\beta+\gamma;c+}f(x) \\
&= {}^{RL}_{c}D^{\beta+\gamma}_{x}\left(\sum_{n=0}^{\infty}\frac{\Gamma(0+n)\omega^n}{\Gamma(0)n!}\,{}^{RL}_{c}I^{\alpha n+\beta+\gamma}_{x}f(x)\right) \\
&= {}^{RL}_{c}D^{\beta+\gamma}_{x}\left({}^{RL}_{c}I^{\beta+\gamma}_{x}f(x)\right) \\
&= f(x),
\end{aligned}$$

where in the final line we used basic properties of Riemann–Liouville differintegral composition from Lemma 1.1.7.

Thus (134) provides a left inverse to the Prabhakar fractional integral, for any value of γ. To check that this expression is independent of γ, we use the series formula (145) again to get:

$$\begin{aligned}
{}^{RL}_{c}D^{\beta+\gamma}_{x}\mathcal{E}^{\omega,-\rho}_{\alpha,\gamma;c+}f(x) &= {}^{RL}_{c}D^{\beta+\gamma}_{x}\left(\sum_{n=0}^{\infty}\frac{\Gamma(-\rho+n)\omega^n}{\Gamma(-\rho)n!}\,{}^{RL}_{c}I^{\alpha n+\gamma}_{x}f(x)\right) \\
&= \sum_{n=0}^{\infty}\frac{\Gamma(-\rho+n)\omega^n}{\Gamma(-\rho)n!}\,{}^{RL}_{c}I^{\alpha n-\beta}_{x}f(x),
\end{aligned}$$

which is independent of γ and precisely equal to the series formula (146) for Prabhakar fractional derivatives. □

Finally, we can prove analogues of the product and chain rule for Prabhakar fractional differintegrals, in the same way as we did for AB differintegrals in §2.4.6.

Theorem 2.5.16. *Let f and g be complex functions such that $f(x)$, $g(x)$, and $f(x)g(x)$ are all in the form $x^{\zeta}\xi(x)$ with $\mathrm{Re}(\zeta) > -1$ and ξ holomorphic on a domain $U \subset \mathbb{C}$. Then for any complex parameters $\alpha, \beta, \omega, \rho, \kappa$ satisfying the conditions* (138)*, the generalised Prabhakar operator satisfies the following version of the product rule:*

$$\mathcal{E}^{\omega,\rho,\kappa}_{\alpha,\beta;c+}\big(f(x)g(x)\big) = \sum_{m=0}^{\infty}\frac{\mathrm{d}^m g}{\mathrm{d}x^m}\left[\sum_{n=0}^{\infty}\frac{\Gamma(\rho+\kappa n)\Gamma(1-\beta-\alpha n)\omega^n}{\Gamma(\rho)\Gamma(1-\beta-\alpha n-m)m!n!}\,{}^{RL}_{c}I^{\alpha n+\beta+m}_{x}f(x)\right]. \tag{151}$$

Proof. Formally, we can argue as follows, using the series formula (139) for generalised

Prabhakar integrals and the Riemann–Liouville version (7) of the product rule:

$$\begin{aligned}\mathcal{E}^{\omega,\rho,\kappa}_{\alpha,\beta;c+}\big(f(x)g(x)\big) &= \sum_{n=0}^{\infty} \frac{\Gamma(\rho+\kappa n)\omega^n}{\Gamma(\rho)n!}\,{}^{RL}_{c}I^{\alpha n+\beta}_x\big(f(x)g(x)\big) \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(\rho+\kappa n)\omega^n}{\Gamma(\rho)n!}\left[\sum_{m=0}^{\infty}\binom{-\alpha n-\beta}{m}\,{}^{RL}_{c}I^{\alpha n+\beta+m}_x f(x)\frac{\mathrm{d}^m g}{\mathrm{d}x^m}\right] \\ &= \sum_{m=0}^{\infty}\frac{\mathrm{d}^m g}{\mathrm{d}x^m}\left[\sum_{n=0}^{\infty} \frac{\Gamma(\rho+\kappa n)\omega^n}{\Gamma(\rho)n!}\binom{-\alpha n-\beta}{m}\,{}^{RL}_{c}I^{\alpha n+\beta+m}_x f(x)\right],\end{aligned}$$

which yields the required expression. To make this proof rigorous, we just need to verify local uniform convergence of the double series found above.

The following finite truncation of (7) can be found as equation (2.199) in [118], and we use it as our starting point:

$${}^{RL}_{c}D^{\alpha}_x\big(f(x)g(x)\big) = \sum_{m=0}^{N}\binom{\alpha}{m}\,{}^{RL}_{c}D^{\alpha-m}_x f(x)\frac{\mathrm{d}^m g}{\mathrm{d}x^m} - R^{\alpha}_N(x), \quad \alpha\in\mathbb{R}, N\geq \alpha+1, \tag{152}$$

where we assume $f\in C[a,x], g\in C^{N+1}[a,x]$, and the remainder term $R^{\alpha}_N(x)$ is defined by

$$R^{\alpha}_N(x) = \frac{1}{N!\Gamma(-\alpha)}\int_c^x (x-y)^{-\alpha-1}f(y)\left[\int_y^x g^{(N+1)}(\xi)(y-\xi)^N\,\mathrm{d}\xi\right]\mathrm{d}y.$$

Here we replace α in (152) by $-\alpha n-\beta$, and substitute the resulting expression into the series formula for generalised Prabhakar integrals:

$$\begin{aligned}\mathcal{E}^{\omega,\rho,\kappa}_{\alpha,\beta;c+}\big(f(x)g(x)\big) &= \sum_{n=0}^{\infty} \frac{\Gamma(\rho+\kappa n)\omega^n}{\Gamma(\rho)n!}\,{}^{RL}_{c}I^{\alpha n+\beta}_x\big(f(x)g(x)\big) \\ &= \sum_{n=0}^{\infty} \frac{\Gamma(\rho+\kappa n)\omega^n}{\Gamma(\rho)n!}\left[\sum_{m=0}^{N}\tbinom{-\alpha n-\beta}{m}\,{}^{RL}_{c}D^{-\alpha n-\beta-m}_x f(x)\frac{\mathrm{d}^m g}{\mathrm{d}x^m} - R^{-\alpha n-\beta}_N(x)\right] \\ &= \sum_{m=0}^{N}\frac{\mathrm{d}^m g}{\mathrm{d}x^m}\left[\sum_{n=0}^{\infty} \frac{\Gamma(\rho+\kappa n)\omega^n}{\Gamma(\rho)n!}\tbinom{-\alpha n-\beta}{m}\,{}^{RL}_{c}I^{\alpha n+\beta+m}_x f(x)\right] \\ &\qquad\qquad - \sum_{n=0}^{\infty} \frac{\Gamma(\rho+\kappa n)\omega^n}{\Gamma(\rho)n!}R^{-\alpha n-\beta}_N(x),\end{aligned}$$

where swapping the sums in the last step is justified because the sum over m is finite and the sum over n is locally uniformly convergent by Theorem 2.5.7. The only thing left to prove now is that

$$\lim_{N\to\infty}\sum_{n=0}^{\infty} \frac{\Gamma(\rho+\kappa n)\omega^n}{\Gamma(\rho)n!}R^{-\alpha n-\beta}_N(x) = 0. \tag{153}$$

To show this, we use an argument similar to that used in §2.4.6. Specifically, equation

(2.201) from [118] tells us that

$$\begin{aligned}&R_N^{-\alpha n-\beta}(x)\\&=\frac{(-1)^N(x-c)^{N+\alpha n+\beta+1}}{N!\Gamma(\alpha n+\beta)}\int_0^1\int_0^1 f\big(c+\eta(x-c)\big)g^{(N+1)}\big(c+(\zeta+\eta-\zeta\eta)(x-c)\big)\,\mathrm{d}\eta\,\mathrm{d}\zeta,\end{aligned}$$

which means that

$$\sum_{n=0}^{\infty}\frac{\Gamma(\rho+\kappa n)\omega^n}{\Gamma(\rho)n!}R_N^{-\alpha n-\beta}(x)=\frac{(-1)^N(x-c)^{N+\beta+1}}{N!}E_{\alpha,\beta}^{\rho,\kappa}\left(\omega(x-c)^{\alpha}\right)\int_0^1\int_0^1[...]\,\mathrm{d}\zeta,$$

the integrand being the same as in the previous expression (which is independent of n). We can then ignore the generalised Mittag-Leffler function when taking the limit, because it is independent of N, and we find that (153) holds just as in [118] and §2.4.6. $\square$

Example 2.5.17. As a simple application of the Prabhakar product rule, we apply the result of Theorem 2.5.16 with $f(x)=e^{ax}$ and $g(x)=x$ and the constant of differintegration $c=i\infty$.

In this case, the outer series in (151) has only two non-trivial terms, namely $m=0$ and $m=1$, while the Riemann–Liouville fractional integral of an exponential function is well known. Thus the series becomes:

$$\begin{aligned}&\sum_{m=0}^{1}\frac{\mathrm{d}^m x}{\mathrm{d}x^m}\left[\sum_{n=0}^{\infty}\frac{\Gamma(\rho+\kappa n)\Gamma(1-\beta-\alpha n)\omega^n}{\Gamma(\rho)\Gamma(1-\beta-\alpha n-m)m!n!}a^{\alpha n+\beta+m}e^{ax}\right]\\&=x\left[\sum_{n=0}^{\infty}\frac{\Gamma(\rho+\kappa n)\Gamma(1-\beta-\alpha n)(\omega a^{\alpha})^n}{\Gamma(\rho)\Gamma(1-\beta-\alpha n)n!}a^{\beta}e^{ax}\right]\\&\qquad\qquad\qquad+\left[\sum_{n=0}^{\infty}\frac{\Gamma(\rho+\kappa n)\Gamma(1-\beta-\alpha n)(\omega a^{\alpha})^n}{\Gamma(\rho)\Gamma(-\beta-\alpha n)n!}a^{\beta+1}e^{ax}\right]\\&=x\left[\sum_{n=0}^{\infty}\frac{\Gamma(\rho+\kappa n)(\omega a^{\alpha})^n}{\Gamma(\rho)n!}a^{\beta}e^{ax}\right]+\left[\sum_{n=0}^{\infty}\frac{\Gamma(\rho+\kappa n)(-\beta-\alpha n)(\omega a^{\alpha})^n}{\Gamma(\rho)n!}a^{\beta+1}e^{ax}\right]\\&=\sum_{n=0}^{\infty}\frac{\Gamma(\rho+\kappa n)(\omega a^{\alpha})^n}{\Gamma(\rho)n!}(x-a\beta-a\alpha n)a^{\beta}e^{ax},\end{aligned}$$

and we have computed the Prabhakar fractional integral $\mathcal{E}_{\alpha,\beta}^{\omega,\rho,\kappa}\left(xe^x\right)$. $\square$

Naturally, Theorem 2.5.16 yields corollaries in the form of product rule analogues for other more specific fractional models, including the already proven (97) for the AB model, by using Proposition 2.5.8.

Theorem 2.5.18. *Let f and g be complex functions such that g is smooth and $f(g(x))$ is a function of the form $x^{\zeta}\xi(x)$ with $\mathrm{Re}(\zeta)>-1$ and ξ holomorphic on a complex domain $U\subset\mathbb{C}$. Then for any complex parameters $\alpha,\beta,\omega,\rho,\kappa$ satisfying the conditions* (138), *the* *generalised Prabhakar operator satisfies the following version of the chain rule:*

$$\mathcal{E}^{\omega,\rho,\kappa}_{\alpha,\beta;c+}\big(f(g(x))\big) = (x-c)^{\beta}\sum_{m=0}^{\infty}\frac{\Gamma(\rho+\kappa m)\Gamma(1-\beta-\alpha m)\,[\omega(x-c)^{\alpha}]^{m}}{\Gamma(\rho)\Gamma(\alpha m+\beta)m!}\sum_{n=0}^{\infty}$$
$$\frac{(c-x)^{n}}{n!(\alpha m+\beta+n)}\left[\sum_{k=1}^{n}\frac{\mathrm{d}^{k}f(g(x))}{\mathrm{d}g(x)^{k}}\sum_{(P_1,\dots,P_n)}\left[\prod_{j=1}^{n}\frac{j}{P_j!(j!)^{P_j}}\left(\frac{\mathrm{d}^{j}g(x)}{\mathrm{d}x^{j}}\right)^{P_j}\right]\right], \quad (154)$$

where the summation over $(P_1,\dots,P_n)$ *is over the set* (9)*.*

Proof. Our starting point is the result of Theorem 2.5.16, which in this case we apply to the product of the two functions $f(g(x))$ and $I(x)=1$. The fractional differintegrals of the unit function I are well known, so the expression (151) becomes:

$$\begin{aligned}
\mathcal{E}^{\omega,\rho,\kappa}_{\alpha,\beta;c+}\big(f(g(x))\big) &= \sum_{n=0}^{\infty}\frac{\mathrm{d}^{n}(f\circ g)}{\mathrm{d}x^{n}}\left[\sum_{m=0}^{\infty}\frac{\Gamma(\rho+\kappa m)\Gamma(1-\beta-\alpha m)\omega^{m}}{\Gamma(\rho)\Gamma(1-\beta-\alpha m-n)m!n!}\,{}^{RL}_{c}I^{\alpha m+\beta+n}_{x}(1)\right]\\
&= \sum_{n=0}^{\infty}\frac{\mathrm{d}^{n}(f\circ g)}{\mathrm{d}x^{n}}\left[\sum_{m=0}^{\infty}\frac{\Gamma(\rho+\kappa m)\Gamma(1-\beta-\alpha m)\omega^{m}}{\Gamma(\rho)\Gamma(1-\beta-\alpha m-n)m!n!}\cdot\frac{(x-c)^{\alpha m+\beta+n}}{\Gamma(\alpha m+\beta+n+1)}\right]\\
&= \sum_{n=0}^{\infty}\frac{\mathrm{d}^{n}(f\circ g)}{\mathrm{d}x^{n}}\left[\sum_{m=0}^{\infty}\frac{\Gamma(\rho+\kappa m)\Gamma(1-\beta-\alpha m)[\omega(x-c)^{\alpha}]^{m}(x-c)^{\beta+n}}{\Gamma(\rho)\frac{\pi(\alpha m+\beta+n)}{\sin(\pi(\alpha m+\beta+n))}m!n!}\right]\\
&= (x-c)^{\beta}\sum_{n=0}^{\infty}\frac{\mathrm{d}^{n}(f\circ g)}{\mathrm{d}x^{n}}\left[\sum_{m=0}^{\infty}\frac{\Gamma(\rho+\kappa m)\Gamma(1-\beta-\alpha m)[\omega(x-c)^{\alpha}]^{m}}{\Gamma(\rho)\frac{\pi(\alpha m+\beta+n)}{\sin(\pi(\alpha m+\beta))}m!}\cdot\frac{(c-x)^{n}}{n!}\right]\\
&= (x-c)^{\beta}\sum_{n=0}^{\infty}\frac{\mathrm{d}^{n}(f\circ g)}{\mathrm{d}x^{n}}\left[\sum_{m=0}^{\infty}\frac{\Gamma(\rho+\kappa m)\Gamma(1-\beta-\alpha m)[\omega(x-c)^{\alpha}]^{m}}{(\alpha m+\beta+n)\Gamma(\rho)\Gamma(\alpha m+\beta)m!}\cdot\frac{(c-x)^{n}}{n!}\right]\\
&= (x-c)^{\beta}\sum_{m=0}^{\infty}\frac{\Gamma(\rho+\kappa m)\Gamma(1-\beta-\alpha m)[\omega(x-c)^{\alpha}]^{m}}{\Gamma(\rho)\Gamma(\alpha m+\beta)m!}\sum_{n=0}^{\infty}\frac{\mathrm{d}^{n}(f\circ g)}{\mathrm{d}x^{n}}\cdot\frac{(c-x)^{n}}{n!(\alpha m+\beta+n)}\\
&= (x-c)^{\beta}\sum_{m=0}^{\infty}\frac{\Gamma(\rho+\kappa m)\Gamma(1-\beta-\alpha m)[\omega(x-c)^{\alpha}]^{m}}{\Gamma(\rho)\Gamma(\alpha m+\beta)m!}\sum_{n=0}^{\infty}\\
&\qquad\frac{(c-x)^{n}}{n!(\alpha m+\beta+n)}\left[\sum_{k=1}^{n}\frac{\mathrm{d}^{k}f(g(x))}{\mathrm{d}g(x)^{k}}\sum_{(P_1,\dots,P_n)}\left[\prod_{j=1}^{n}\frac{j}{P_j!(j!)^{P_j}}\left(\frac{\mathrm{d}^{j}g(x)}{\mathrm{d}x^{j}}\right)^{P_j}\right]\right],
\end{aligned}$$

as required, where we used the classical Faà di Bruno formula in the final step. □

Once again, Theorem 2.5.18 together with Proposition 2.5.8 yields analogues of the chain rule for other more specific fractional models, including the already established result (100) for the AB model, as well as a similar result for the Caputo–Fabrizio model.

One application of Theorem 2.5.18 would be to compute fractional differintegrals of a Gaussian function e^{-x^2}, by putting $f(x)=e^{x}$ and $g(x)=-x^{2}$ in the identity (154).

2.6 A fractional formula for the Lerch zeta function

2.6.1 Introduction

The idea of combining fractional calculus and analytic number theory was born in the work of Keiper, who in his 1975 MSc thesis [85] established a formula for the Riemann zeta function as a Riemann–Liouville fractional derivative. It has only been revived very recently, in the work of Guariglia et al [71, 31, 32] and also Srivastava et al [93, 94, 132]. But the Guariglia papers use a different model of fractional calculus, namely a recent variant due to Ortigueira of the Caputo model, while the Srivastava papers only consider fractional expressions for generalisations of the Lerch zeta function in terms of each other, not in terms of elementary functions.

Here we establish a new relationship between fractional calculus and zeta functions, by writing the Lerch zeta function as a fractional derivative of a much simpler function. We use only the classical Riemann–Liouville model of fractional calculus, without the extra complications introduced by newer models. For this chapter we shall denote the Lerch zeta function, defined in (17), as

$$L(t,x,s)=\sum_{n=0}^{\infty}(n+x)^{-s}e^{2\pi itn},\qquad \mathrm{Re}(s)>1,\mathrm{Re}(x)>0,\mathrm{Im}(t)\geq 0.$$

The variable t here is *not* the imaginary part of s, which is often denoted by t in the literature on zeta functions; it is an entirely independent variable. We use the notation t instead of λ only because it seems natural when we differentiate with respect to this variable later on.

Note that differentiation with respect to t explains why we must necessarily use the Lerch zeta function rather than the simpler Hurwitz or Riemann zeta functions: the third parameter t in $L(t,x,s)$ plays a vital role in our derivation.

Of course, our result (155), expressing the Lerch zeta function as a fractional derivative of an elementary function, yields a similar expression for the Riemann zeta function too, simply by setting the values of x and t appropriately. Thus, we lose nothing by starting from the more general Lerch zeta function instead of the Riemann zeta function.

The following statement concerning fractional integration of series will be used later on in the proof of the main result.

Lemma 2.6.1. *If the series $f(t)=\sum_{n=1}^{\infty}f_n(t)$ is uniformly convergent on a complex disc $|t-c|\leq R$ with $c-R\notin\mathbb{R}_0^+$, and the constants δ, α satisfy $\delta>0$, $\mathrm{Re}(\alpha)<0$, and*

$$\left[\sum_{n=N+1}^{\infty}f_n(t)\right]t^{\delta-\alpha}\to 0 \text{ as } N\to\infty$$

uniformly on the ray from $c - R$ to negative infinity, then we have

$$ {}_{-\infty}D_t^{\alpha} f(t) = \sum_{n=1}^{\infty} {}_{-\infty}D_t^{\alpha} f_n(t) $$

for $|t - c| \leq R$, and the series of fractional integrals is locally uniformly convergent.

Proof. This result is established by the proof of [85, Theorem IX]. (In that proof, it was assumed that c is real, but this was only for convenience – the same argument works for complex c provided that $c - R \notin \mathbb{R}_0^+$.) $\square$

2.6.2 The main result

The crux of this chapter is the following theorem expressing the Lerch zeta function as a fractional differintegral.

Theorem 2.6.2. *The Lerch zeta function can be written as*

$$ L(t,x,s) = (2\pi)^s \exp\left[i\pi(\tfrac{s}{2} - 2tx)\right] {}_{-\infty}D_t^{-s} \left(\frac{e^{2\pi i t x}}{1 - e^{2\pi i t}} \right) \tag{155} $$

for any complex numbers s, x, t satisfying $\mathrm{Im}(t) > 0$ *and* $x \notin (-\infty, 0]$.

Proof. We start from the definition (17) of the Lerch zeta function, and use the result of Lemma 1.1.3 to rewrite the summand as a fractional differintegral:

$$ \begin{aligned} L(t,x,s) = \sum_{n=0}^{\infty} (n+x)^{-s} e^{2\pi i t n} &= (2\pi i)^s e^{-2\pi i t x} \sum_{n=0}^{\infty} (2\pi i)^{-s} (n+x)^{-s} e^{2\pi i t (n+x)} \\ &= (2\pi i)^s e^{-2\pi i t x} \sum_{n=0}^{\infty} {}_{-\infty}D_t^{-s} \left(e^{2\pi i t (n+x)} \right). \end{aligned} \tag{156} $$

So far our argument is valid for all $t, x, s \in \mathbb{C}$ such that $\mathrm{Re}(s) > 1$, $\mathrm{Re}(x) > 0$, and $\mathrm{Im}(t) \geq 0$. These conditions come from the series definition (17); the extra condition that $2\pi i(n + x) \notin \mathbb{R}_0^-$, required by Lemma 1.1.3, is automatically satisfied for all $n \geq 0$ due to the condition we already have on x. Note that since s has positive real part, the fractional operator appearing in (156) is an integral and not a derivative.

The next consideration is whether or not the summation and fractional integration operators in (156) can be swapped. For any $\epsilon > 0$, the series

$$ \sum_{n=0}^{\infty} e^{2\pi i t (n+x)} $$

converges uniformly on the closed region $\mathrm{Im}(t) \geq \epsilon$ of the upper half t-plane, and indeed

$$\left(\sum_{n=N+1}^{\infty} e^{2\pi it(n+x)}\right) t^{\delta-s} \to 0 \text{ as } N \to \infty$$

uniformly on this region for any fixed $\delta < 1$. So, under the slightly strengthened condition $\mathrm{Im}(t) > 0$, it follows from Lemma 2.6.1 that the series of fractional integrals also converges locally uniformly and

$$_{-\infty}D_t^{-s}\left(\sum_{n=0}^{\infty} e^{2\pi it(n+x)}\right) = \sum_{n=0}^{\infty} {}_{-\infty}D_t^{-s}\left(e^{2\pi it(n+x)}\right).$$

Substituting this identity into the expression (156) yields:

$$\begin{aligned}
L(t,x,s) &= (2\pi i)^s e^{-2\pi itx}\,{}_{-\infty}D_t^{-s}\left(\sum_{n=0}^{\infty} e^{2\pi it(n+x)}\right) \\
&= (2\pi i)^s e^{-2\pi itx}\,{}_{-\infty}D_t^{-s}\left(e^{2\pi itx}\sum_{n=0}^{\infty}\left(e^{2\pi it}\right)^n\right) \\
&= (2\pi i)^s e^{-2\pi itx}\,{}_{-\infty}D_t^{-s}\left(\frac{e^{2\pi itx}}{1-e^{2\pi it}}\right) \\
&= (2\pi)^s \exp\left[i\pi(\tfrac{s}{2}-2tx)\right]\,{}_{-\infty}D_t^{-s}\left(\frac{e^{2\pi itx}}{1-e^{2\pi it}}\right),
\end{aligned}$$

as required.

We have now proved the main result (155) under the following assumptions:

$$\mathrm{Re}(s) > 1, \quad \mathrm{Re}(x) > 0, \quad \mathrm{Im}(t) > 0.$$

By analytic continuation, these assumptions can be relaxed to any $t, x, s \in \mathbb{C}$ such that both sides of (155) are still holomorphic. We know from [90, Theorem 2.3] that the left-hand side $L(t,x,s)$ can be extended to a holomorphic function on the domain

$$\{(t,x,s) \in \mathbb{C}\times\mathbb{C}\times\mathbb{C} : \mathrm{Im}(t) > 0, x \notin (-\infty, 0]\},$$

this domain being embeddable into the universal cover of $(\mathbb{C}\backslash\mathbb{Z}) \times (\mathbb{C}\backslash\mathbb{Z}_0^-) \times \mathbb{C}$.

The right-hand side of (155) is clearly going to be holomorphic in x wherever it is well-defined, and ditto in s by [127, §2.4]. It is well-defined and holomorphic in t provided that the fractional differintegral is well-defined and holomorphic in t.

For $\mathrm{Re}(s) > 0$, this differintegral can be written as

$$\tfrac{1}{\Gamma(s)}\int_{-\infty}^{t} (t-u)^{s-1}\frac{e^{2\pi iux}}{1-e^{2\pi iu}}\,\mathrm{d}u. \tag{157}$$

The integrand here is holomorphic in $u \in \mathbb{C}\backslash\mathbb{Z}$, since we are assuming the contour of integration to be horizontal in the complex plane. Thus the whole expression is well-defined and holomorphic for any t, x, s such that $\mathrm{Im}(t) > 0$ and the integral converges at both endpoints.

Near $u = t$, the exponential-fraction part of the integrand is constant, so the integral behaves like $(t-u)^s$, which converges since we have assumed $\mathrm{Re}(s) > 0$.

Near $u = -\infty$, the exponential denominator is bounded (since we have $\mathrm{Im}(u) > 0$), the numerator has exponential decay provided that $\mathrm{Im}(x) < 0$, and the $(t-u)^{s-1}$ term has only polynomial growth.

Thus the expression (157) is well-defined and holomorphic in all three variables provided that $\mathrm{Re}(s) > 0$, $\mathrm{Im}(x) < 0$, and $\mathrm{Im}(t) > 0$.

We can extend the region of validity to cover $x \in \mathbb{R}\backslash\mathbb{Z}_0^-$ too, given an extra restriction on s. The series

$$\sum_{n=0}^{\infty} e^{2\pi iu(x+n)} = \frac{e^{2\pi iux}}{1-e^{2\pi iu}}$$

is uniformly convergent, since u has a fixed positive imaginary part. Therefore (157) can be rewritten, regardless of x, in the form of the series

$$\tfrac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \int_{-\infty}^{t} (t-u)^{s-1} e^{2\pi iu(x+n)} \, \mathrm{d}u,$$

whose integral summand is well-defined for $x \in \mathbb{R}\backslash\mathbb{Z}_0^-$ provided that $\mathrm{Re}(s) \leq 1$.

Given the definition (2) of fractional derivatives, the $\mathrm{Re}(s) > 0$ requirement can be eliminated immediately.

So the main result (155) is now proved under the following assumption:

$$\mathrm{Im}(x) < 0, \mathrm{Im}(t) > 0 \quad \text{ or } \quad \mathrm{Re}(s) \leq 1, x \in \mathbb{R} \setminus \mathbb{Z}_0^-, \mathrm{Im}(t) > 0.$$

But we already know that (155) is also valid for $\mathrm{Re}(s) > 1, \mathrm{Re}(x) > 0, \mathrm{Im}(t) > 0$. Thus, by taking unions of domains, we can say that it is always valid for

$$\mathrm{Im}(x) < 0 \text{ or } x \in \mathbb{R}^+, \qquad \mathrm{Im}(t) > 0.$$

Finally, it is clear from the definition (17) that the Lerch zeta function satisfies the following basic functional equation:

$$\overline{L(t,x,s)} = L(-\bar{t}, \bar{x}, \bar{s}). \tag{158}$$

The condition $\mathrm{Im}(t) > 0$ is preserved by mapping t to $-\bar{t}$, but if $\mathrm{Im}(x) \leq 0$, then $\mathrm{Im}(\bar{x}) \geq 0$. Thus, if (155) is known to be valid for the lower half plane part of $\mathbb{C}\backslash\mathbb{R}_0^-$, then by taking complex conjugates it follows that it is also valid for the upper half plane part of $\mathbb{C}\backslash\mathbb{R}_0^-$, and therefore for all $x \in \mathbb{C}\backslash\mathbb{R}_0^-$. □

Remark 2.6.3. Note that unlike previous results on the fractional calculus of zeta functions [85, 71], our formula depends crucially on using the Lerch zeta function rather than the Riemann or Hurwitz zeta functions. The third parameter t – i.e. the one which appears in the Lerch function $L(t, x, s)$ but not the Riemann or Hurwitz functions – is a fundamental part of our result (155): we could not have achieved analogous results for $\zeta(s)$ or $\zeta(x, s)$ without first introducing this extra parameter in order to differentiate with respect to it.

It is, however, possible to obtain a formula for the Riemann zeta function as a corollary of Theorem 2.6.2, as follows.

Corollary 2.6.4. *The Riemann zeta function can be written as*

$$\zeta(s) = \frac{(2\pi i)^s}{2^{1-s} - 1} \, {}_{-\infty}D_t^{-s} \left(\frac{1}{e^{-2\pi i t} - 1} \right) \Bigg|_{t=\frac{1}{2}} \tag{159}$$

for any $s \in \mathbb{C}$, *or alternatively as*

$$\zeta(s) = (2\pi i)^s \lim_{t \to 0} \left({}_{-\infty}D_t^{-s} \left(\frac{1}{e^{-2\pi i t} - 1} \right) \right) \tag{160}$$

for $\mathrm{Re}(s) > 1$.

Proof. The first identity (159) follows by letting $t = \frac{1}{2}$ in (155) and noting the fact that

$$L(\tfrac{1}{2}, 1, s) = (1 - 2^{1-s})\zeta(s).$$

The second identity (160) follows by letting $t \to 0$ in (155) and recalling the series definitions (15), (17). We note that (160) does not hold in general, because the limit as $t \to 0$ of the Lerch function does not always exist [106]. □

2.6.3 Further discussion and corollaries

Remark 2.6.5. We verify that our new formula satisfies the complex conjugation relation (158) for the Lerch zeta function. Using the right-hand side of (155) as the definition of $L(t, x, s)$, we get:

$$\begin{aligned}
\overline{L(t, x, s)} &= (2\pi)^{\bar{s}} \exp\left[-i\pi(\tfrac{\bar{s}}{2} - 2\bar{t}\bar{x})\right] \overline{{}_{-\infty}D_{u=t}^{-s} \left(\frac{e^{2\pi i u x}}{1 - e^{2\pi i u}} \right)} \\
L(-\bar{t}, \bar{x}, \bar{s}) &= (2\pi)^{\bar{s}} \exp\left[i\pi(\tfrac{\bar{s}}{2} + 2\bar{t}\bar{x})\right] {}_{-\infty}D_{u=-\bar{t}}^{-\bar{s}} \left(\frac{e^{2\pi i u \bar{x}}}{1 - e^{2\pi i u}} \right)
\end{aligned}$$

(We use the notation $D^{\alpha}_{u=t}f(u)$ instead of $D^{\alpha}_{t}f(t)$ in order to avoid confusion in the case where t is replaced by $-\bar{t}$.) Thus, to verify (158) it will be sufficient to show that

$$\overline{{}_{-\infty}D^{-s}_{u=t}\left(\frac{e^{2\pi iux}}{1-e^{2\pi iu}}\right)} = e^{i\pi s}\,{}_{-\infty}D^{-\bar{s}}_{u=-\bar{t}}\left(\frac{e^{2\pi iu\bar{x}}}{1-e^{2\pi iu}}\right),$$

or in other words, assuming $\mathrm{Re}(s) > 0$,

$$\overline{\tfrac{1}{\Gamma(\bar{s})}\int_{-\infty}^{t}(t-u)^{s-1}\frac{e^{2\pi iux}}{1-e^{2\pi iu}}\,\mathrm{d}u} = e^{i\pi s}\tfrac{1}{\Gamma(\bar{s})}\int_{-\infty}^{-\bar{t}}(-\bar{t}-u)^{\bar{s}-1}\frac{e^{2\pi iu\bar{x}}}{1-e^{2\pi iu}}\,\mathrm{d}u.$$

Writing $t = a+bi$ and $-\bar{t} = -a+bi$ and $u = r+bi$, this becomes

$$\int_{-\infty}^{a}(a-r)^{\bar{s}-1}\frac{e^{-2\pi i\bar{u}\bar{x}}}{1-e^{-2\pi i\bar{u}}}\,\mathrm{d}r = e^{i\pi s}\int_{-\infty}^{-a}(-a-r)^{\bar{s}-1}\frac{e^{2\pi iu\bar{x}}}{1-e^{2\pi iu}}\,\mathrm{d}r.$$

Since $b > 0$, both denominators can be expanded as series, so it is sufficient to prove that

$$\int_{-\infty}^{a}(a-r)^{\bar{s}-1}e^{-2\pi i\bar{u}(\bar{x}+n)}\,\mathrm{d}r = e^{i\pi s}\int_{-\infty}^{-a}(-a-r)^{\bar{s}-1}e^{2\pi iu(\bar{x}+n)}\,\mathrm{d}r$$

for all $n \in \mathbb{Z}^+_0$. Making a linear substitution and factoring out constant terms, this reduces to

$$\int_{0}^{\infty}p^{\bar{s}-1}e^{2\pi ip(\bar{x}+n)}\,\mathrm{d}p = e^{i\pi s}\int_{0}^{\infty}p^{\bar{s}-1}e^{-2\pi ip(\bar{x}+n)}\,\mathrm{d}p,$$

or equivalently

$$\int_{0}^{\infty}p^{\bar{s}-1}e^{2\pi ip(\bar{x}+n)}\,\mathrm{d}p = -\int_{-\infty}^{0}p^{\bar{s}-1}e^{2\pi ip(\bar{x}+n)}\,\mathrm{d}p,$$

where the integral along the negative real axis is assumed to be with argument $+\pi$. And by Jordan's lemma, closing the real contour in the upper half plane gives

$$\int_{-\infty}^{\infty}p^{\bar{s}-1}e^{2\pi ip(\bar{x}+n)}\,\mathrm{d}p = 0$$

for all $n \geq 0$, provided that $x \in \mathbb{R}^+$ and $\mathrm{Re}(s) < 1$.

So we have re-verified the identity (158) under the assumptions $0 < \mathrm{Re}(s) < 1, x \in \mathbb{R}^+, \mathrm{Im}(t) > 0$. This acts as a confirmation of the correctness of our result.

The result of Theorem 2.6.2 is an expression for the Lerch zeta function as the product of a fractional differintegral and a simple explicit term. We now demonstrate how this explicit term arises naturally from consideration of the Lerch zeta function and its properties, and thence derive a second formula for the Lerch zeta function in terms of fractional differintegrals.

Remark 2.6.6. It is known [11, 89, 92] that for $s, t \in \mathbb{C}$ with $\mathrm{Im}(t) > 0$ and $x \in (0, 1)$, or with $t, x \in (0,1)$, the Lerch zeta function satisfies the following functional equation:

$$L(t,x,1-s) = \frac{\Gamma(s)}{(2\pi)^s}\Big(\exp\left[i\pi(\tfrac{s}{2}-2tx)\right] L(-x,t,s) \\ + \exp\left[-i\pi(\tfrac{s}{2}-2x(1-t))\right] L(x,1-t,s)\Big) \quad (161)$$

Thus, we observe that the exponential multiplier term $\exp\left[i\pi(\frac{s}{2}-2tx)\right]$ seen in (155) is already known to arise from essential properties of the Lerch zeta function. This demonstrates the naturality of the result of Theorem 2.6.2.

Theorem 2.6.7. *The Lerch zeta function can be written as*

$$L(t,x,1-s) = \Gamma(s)e^{i\pi s}\,{}_{-\infty}D_u^{-s}\left(\frac{e^{2\pi itu}}{1-e^{-2\pi iu}}\right)\bigg|_{u=-x} - \Gamma(s)\,{}_{-\infty}D_x^{-s}\left(\frac{e^{-2\pi itx}}{1-e^{2\pi ix}}\right) \quad (162)$$

where s, x, t are any complex numbers satisfying $\mathrm{Im}(t) > 0$ *and* $x \in (0,1)$.

Proof. In order to use the identity (161) together with the new expression (155), we will need to show that (155) can be extended from $\mathrm{Im}(t) > 0$ to the line $t \in \mathbb{R}\backslash\mathbb{Z}$. This can be proved by continuity, provided that we choose the right contour for the integration inherent in the fractional differintegral. When $t \in \mathbb{R}\backslash\mathbb{Z}$, the straight ray from t to $-\infty$ contains infinitely many poles of the function $\frac{e^{2\pi itx}}{1-e^{2\pi it}}$, so the integral must be defined as a limit:

$${}_{-\infty}D_t^{-s}\left(\frac{e^{2\pi itx}}{1-e^{2\pi it}}\right) = \lim_{\epsilon\to 0^+}\left[\frac{1}{\Gamma(s)}\int_{-\infty}^{t'}(t'-u)^{s-1}\frac{e^{2\pi iux}}{1-e^{2\pi iu}}\,\mathrm{d}u\right]_{t'=t+i\epsilon}, \quad t \in \mathbb{R}. \quad (163)$$

With this definition, it is clear by continuity that (155) holds for all t with $\mathrm{Im}(t) \geq 0$ and $t \notin \mathbb{Z}$.

Now we can start from the functional equation (161) and substitute (155) for the two Lerch functions on the right-hand side. For simplicity, we shall drop the left-subscript $-\infty$ on the fractional operators, since they all use the same constant of differintegration. We also use the notation $D^\alpha_{u=x}f(u)$ instead of $D^\alpha_x f(x)$, just to avoid confusion in the case where x is replaced by $-x$.

$$
\begin{aligned}
L(t,x,1-s) &= \frac{\Gamma(s)}{(2\pi)^s}\Big(\exp\left[i\pi(\tfrac{s}{2}-2tx)\right] L(-x,t,s) \\
&\qquad\qquad\qquad + \exp\left[-i\pi(\tfrac{s}{2}-2x(1-t)\right] L(x,1-t,s)\Big) \\
&= \frac{\Gamma(s)}{(2\pi)^s}\Bigg(\exp\left[i\pi(\tfrac{s}{2}-2tx)\right](2\pi)^s \exp\left[i\pi(\tfrac{s}{2}+2tx)\right] D^{-s}_{u=-x}\left(\frac{e^{2\pi itu}}{1-e^{2\pi iu}}\right) \\
&\quad + \exp\left[-i\pi(\tfrac{s}{2}-2x(1-t)\right](2\pi)^s \exp\left[i\pi(\tfrac{s}{2}-2(1-t)x)\right] D^{-s}_{u=x}\left(\frac{e^{2\pi i(1-t)u}}{1-e^{2\pi iu}}\right)\Bigg) \\
&= \Gamma(s)\Bigg(\exp\left[i\pi s\right] D^{-s}_{u=-x}\left(\frac{e^{2\pi itu}}{1-e^{2\pi iu}}\right) + D^{-s}_{u=x}\left(\frac{e^{2\pi i(1-t)u}}{1-e^{2\pi iu}}\right)\Bigg) \\
&= \Gamma(s)\Bigg(e^{i\pi s} D^{-s}_{u=-x}\left(\frac{e^{2\pi itu}}{1-e^{2\pi iu}}\right) - D^{-s}_{u=x}\left(\frac{e^{-2\pi itu}}{1-e^{-2\pi iu}}\right)\Bigg).
\end{aligned}
$$

And the required result follows. □

Remark 2.6.8. The results of Theorems 2.6.2 and 2.6.7 can be used to provide a new elementary proof of the functional equation (161).

In the proof of Theorem 2.6.7, we used the new expression (155) for the Lerch zeta function to reduce the right-hand side of the functional equation (161) to an expression in terms of two fractional differintegrals. If we can rewrite this expression using elementary methods as simply $L(t,x,1-s)$, then we have rederived the functional equation using fractional calculus.

Therefore, we start from the right-hand side of (162) and proceed as follows:

$$
\begin{aligned}
(\text{RHS of (162)}) &= e^{i\pi s}\int_{-\infty}^{-x}(-x-u)^{s-1}\frac{e^{2\pi itu}}{1-e^{-2\pi iu}}\,\mathrm{d}u - \int_{-\infty}^{x}(x-u)^{s-1}\frac{e^{-2\pi itu}}{1-e^{2\pi iu}}\,\mathrm{d}u \\
&= e^{i\pi s}\int_{x}^{\infty}(-x+u)^{s-1}\frac{e^{-2\pi itu}}{1-e^{2\pi iu}}\,\mathrm{d}u - \int_{-\infty}^{x}(x-u)^{s-1}\frac{e^{-2\pi itu}}{1-e^{2\pi iu}}\,\mathrm{d}u \\
&= -\int_{-\infty}^{\infty}(x-u)^{s-1}\frac{e^{-2\pi itu}}{1-e^{2\pi iu}}\,\mathrm{d}u,
\end{aligned}
$$

where the contour of integration from $-\infty$ to $+\infty$ crosses the real axis at x, passing above all the poles to the left of x and below all the poles to the right of x. This choice of contour follows from the definition given by (163).

By Jordan's lemma, for $t \in \mathbb{R}$ and $\mathrm{Re}(s) < 1$, the contour can be closed in the lower half plane. Then the residue theorem yields

$$
\begin{aligned}
(\text{RHS of } (162)) &= 2\pi i \sum_{n=0}^{\infty} \mathrm{Res}_{u=-n} \left((x-u)^{s-1} \frac{e^{-2\pi itu}}{1-e^{2\pi iu}} \right) \\
&= 2\pi i \sum_{n=0}^{\infty} (x+n)^{s-1} \frac{e^{2\pi itn}}{2\pi i} \\
&= \sum_{n=0}^{\infty} (x+n)^{s-1} e^{2\pi itn} = L(t, x, 1-s),
\end{aligned}
$$

as required. Thus we have proved the functional equation (161) in the case where $0 < x < 1, 0 < t < 1, \mathrm{Re}(s) < 1$.

To conclude: in this chapter we have forged a new connection between fractional calculus and the theory of zeta functions. This connection is different from others that have previously been discovered: it was found by using the Lerch zeta function, a significant generalisation of the more commonly seen Riemann and Hurwitz zeta functions, and it enables all of these zeta functions to be expressed as standard fractional derivatives of elementary functions. We have also demonstrated the usefulness of our result by indicating its natural interplay with fundamental properties of zeta functions, and how it can even be used to provide new proofs of some of these properties.

Any new formula for zeta functions is potentially useful, as it gives a new angle of attack in the ceaseless attempts to establish important properties of such functions. It is especially important to establish more links between fractional calculus and analytic number theory, in order to increase the probability that all the machinery of one field can be brought to bear on the problems of the other.

The formulae proved here could be just the start of a whole new project bringing together two distinct fields of study. For example, basic theorems of fractional calculus may now be usable to generate significant new expressions for zeta functions. Creating new links between different areas is always an opportunity, and this is surely no exception.

Part 3

Asymptotics for zeta functions

This Part of the manuscript contains my research on asymptotics related to zeta functions, as indicated briefly in §1.2 of the introduction. A more detailed summary is as follows.

- §3.1 concerns the large-t asymptotics to all orders of the Hurwitz zeta function $\zeta(x, \sigma + it)$. The main result is Theorem 3.1.19. The subchapters are §3.1.1 to provide background and motivation, §3.1.2 to derive an exact integral formula for the Hurwitz zeta function, §3.1.3–§3.1.5 to analyse the asymptotic behaviour to all orders of each integral in this exact formula, and §3.1.6 to conclude with the final result and some remarks and comparisons to previous related work.

- §3.2 concerns the uniform asymptotics of a particular integral expression in the neighbourhood of a stationary point. The main result is Theorem 3.2.5. The subchapters are §3.2.1 to set up the problem and provide surrounding context, §3.2.2 to present the integral splitting which is the starting point of the analysis, §3.2.3 and §3.2.4 to analyse the asymptotic behaviour of each constituent part of this split integral, and §3.2.5 to conclude with the final result and some remarks and comparisons to previous related work.

3.1 Asymptotics to all orders of the Hurwitz zeta function

3.1.1 Introduction

For this work, we shall slightly modify the Hurwitz zeta function in order to get an easier analysis. The **modified Hurwitz zeta function**, denoted by $\zeta_1(x,s)$, is defined on a right half plane by

$$\zeta(x,s) \coloneqq \sum_{n=1}^{\infty}(n+x)^{-s}, \quad \mathrm{Re}(x) > -1, s = \sigma + it, \sigma > 1, t \in \mathbb{R}, \tag{164}$$

and as usual defined for all $s \in \mathbb{C}\backslash\{1\}$ by analytic continuation. It is clear that this is related to the Hurwitz zeta function by the following simple formulae:

$$\zeta_1(x,s) = \zeta(x,s) - \frac{1}{x^s};$$
$$\zeta_1(x,s) = \zeta(x+1,s);$$

and that for $x = 0$ the modified Hurwitz function reduces to the Riemann zeta function:

$$\zeta(s) = \zeta_1(x,s).$$

The reason for using this very minor modification of the Hurwitz zeta function is partly to simplify some of the calculation below, which would otherwise be using $x+1$ instead of x, and partly so that the reduction to the Riemann zeta function is at $x = 0$ rather than $x = 1$, meaning that for 'small' x the function approximates to the Riemann case.

We already saw in §1.2 the approximate functional equation (18) for the Riemann zeta function. The analogous formula for the modified Hurwitz function is the following asymptotic expression, proved in e.g. [124]:

$$\zeta_1(\alpha,s) = \sum_{n=1}^{\lfloor x-\alpha \rfloor}(n+\alpha)^{-s} + \chi(s)\sum_{n=1}^{y}\frac{\sin\left(\frac{\pi s}{2}+2\pi n\alpha\right)}{\sin\left(\frac{\pi s}{2}\right)}n^{s-1} + O\big(x^{-\sigma}\log(y+2) + x^{1-\sigma}t^{-1/2}\big), \tag{165}$$

where

$$xy = \tfrac{t}{2\pi}, 0 < \sigma < 1, 0 < \alpha \leq 1, t \to \infty,$$

and once again the entire function χ is defined by (19). A similar, more general expression for the Lerch zeta function was proved in [65].

We wish to improve the first-order asymptotic formula (165) in order to establish asymptotics to all orders for the modified Hurwitz zeta function. In order to do this, we shall follow roughly the same methodology as Fokas and Lenells [57], who proved asymptotics to all orders for the Riemann zeta function. Our starting point is the following exact formula, analogous to their formula (20) for the Riemann zeta function:

$$\zeta_1(x,s) = \chi(s)\Bigg(\sum_{m=1}^{\lfloor \eta/2\pi \rfloor} e^{-2\pi i m x} m^{s-1} - \tfrac{e^{-i\pi s/2}}{(2\pi)^s}\int_{\hat{C}_\eta^0} \frac{e^{(1+x)z} - e^{-xz}}{1-e^z} z^{s-1}\,\mathrm{d}z \\ + \tfrac{e^{i\pi s/2}}{(2\pi)^s}\int_{-i\eta}^{\infty e^{i\phi_2}} \frac{e^{-(1+x)z}}{1-e^{-z}} z^{s-1}\,\mathrm{d}z + \tfrac{e^{-i\pi s/2}}{(2\pi)^s}\int_{i\eta}^{\infty e^{i\phi_1}} \frac{e^{-(1+x)z}}{1-e^{-z}} z^{s-1}\,\mathrm{d}z\Bigg), \quad (166)$$

valid for

$$0 < \eta < \infty, -\tfrac{\pi}{2} < \phi_1, \phi_2 < \tfrac{\pi}{2}, 0 < \sigma \leq 1, 0 < t < \infty, 0 < x < \infty,$$

where the contour $\hat{C}_\eta^0$ is defined in Definition 3.1.5 below. We prove the identity (166) in §3.1.2 below, and then analyse it using an integration by parts method as seen in [103], eventually deriving an expression for the large-t asymptotics of $\zeta_1(x,s)$ to all orders.

Remark 3.1.1. We note the following comparisons between our analysis and the analysis in [57].

1. Equation (20) suggests separate analysis for the cases $t < \eta, t = \eta, t > \eta$. These three cases were indeed analysed separately in [57], but in our approach, we present a unified treatment. Our analysis requires a certain condition to be placed on η, but this condition is not very restrictive.

2. The asymptotic estimation of certain integrals appearing in [57] led to their analysis via the stationary point technique. Here, by rewriting such integrals in terms of integrals which can be computed explicitly and integrals which do *not* include stationary points, we have avoided the stationary point analysis.

3. The representations presented in [57] involve a finite series for the case of $\eta < t$ but an infinite series for the case of $\eta \geq t$. Since our approach for all values of η is analogous to that used in [57] for the case of $\eta \geq t$, we first derive a representation which involves an infinite series. However, we are then able to replace this infinite series by a finite one, some of whose upper bounds depend on η. Thus, our final result is analogous to that of [57] in the case of $\eta < t$, since it is a finite series, but it is less uniform, in the sense that the length of this finite series depends on η.

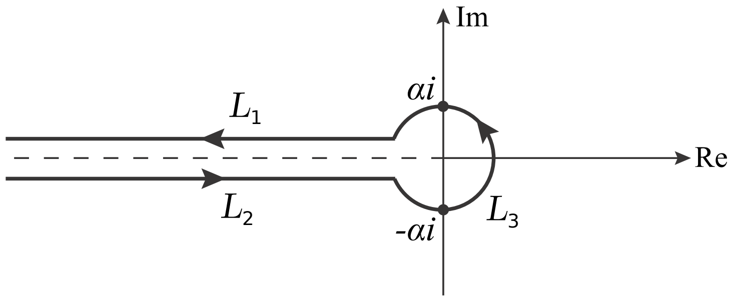

Figure 3: The contours L_1, L_2, L_3 which together form the Hankel contour H_α

3.1.2 An explicit formula

Definition 3.1.2. The Hankel contour H_α is defined by the following three components:

$$\begin{aligned}
L_1 &= \{\alpha e^{i\theta} : \tfrac{\pi}{2} < \theta < \pi\} \cup \{re^{i\pi} : \alpha < r < \infty\},\\
L_2 &= \{re^{-i\pi} : \alpha < r < \infty\} \cup \{\alpha e^{i\theta} : -\pi < \theta < -\tfrac{\pi}{2}\},\\
L_3 &= \{\alpha e^{i\theta} : -\tfrac{\pi}{2} < \theta < \tfrac{\pi}{2}\},
\end{aligned}$$

where α is a constant with $0 < \alpha < 2\pi$. We define H_α to be the union of the L_j, as shown in Figure 3.

The following lemma is a standard one, proved e.g. in [12, Chapter 12], but we include its proof to introduce the methodology of complex contour integrals that we are using here.

Lemma 3.1.3. *The meromorphic continuation of the modified Hurwitz zeta function to all $s \in \mathbb{C}$ is given by*

$$\zeta_1(x, s) = \frac{\Gamma(1-s)}{2\pi i} \int_{H_\alpha} \frac{e^{xz} z^{s-1}}{e^{-z} - 1} \, \mathrm{d}z, \tag{167}$$

for $\mathrm{Re}(x) > -1$*, where H_α is the Hankel contour defined by Definition 3.1.2.*

Proof. For any $n \in \mathbb{N}$ and $s, x \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(x) > -1$, we have

$$\frac{1}{(n+x)^s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-(n+x)z} z^{s-1} \, \mathrm{d}z.$$

Taking the sum over all $n \in \mathbb{N}$, and using the fact that $\sum_{n=1}^\infty e^{-nz} = \frac{1}{e^z - 1}$ is a locally uniformly convergent series for $\mathrm{Re}(z) > 0$, we find

$$\zeta_1(x, s) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-xz} z^{s-1} \sum_{n=1}^\infty e^{-nz} \, \mathrm{d}z = \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-xz} z^{s-1}}{e^z - 1} \, \mathrm{d}z \tag{168}$$

for $\mathrm{Re}(s) > 1$ and $\mathrm{Re}(x) > -1$. In what follows we shall show that the right-hand sides of equations (167) and (168) are identical and that the former is meromorphic for all $s, x \in \mathbb{C}$ with $\mathrm{Re}(x) > -1$; this will suffice to establish the required result.

For $\text{Re}(s) > 1$, we can let $\alpha \to 0$ in the Hankel-contour formula, so that the integral around the curved part L_3 of the contour can be computed using Cauchy's theorem:

$$\int_{L_3} \frac{e^{xz} z^{s-1}}{e^{-z} - 1} \,\mathrm{d}z = \lim_{\alpha \to 0} \left(\int_{-\pi/2}^{\pi/2} \frac{e^{xz} z^{s}}{e^{-z} - 1} \bigg|_{z=\alpha e^{i\theta}} \mathrm{d}\theta \right) = \lim_{\alpha \to 0} \pi \left(\frac{z^s}{-z} \right) \bigg|_{z=\alpha e^{i\theta}} = 0.$$

Hence, the Hankel contour integral expression yields:

$$\begin{aligned}
\frac{\Gamma(1-s)}{2\pi i} \int_{H_\alpha} \frac{e^{xz} z^{s-1}}{e^{-z} - 1} \,\mathrm{d}z &= \frac{\Gamma(1-s)}{2\pi i} \left(\int_0^\infty \frac{e^{-xu} u^{s-1} e^{i\pi s}}{e^u - 1} \,\mathrm{d}u - \int_0^\infty \frac{e^{-xu} u^{s-1} e^{-i\pi s}}{e^u - 1} \,\mathrm{d}u \right) \\
&= \frac{\Gamma(1-s) \sin(\pi s)}{\pi} \int_0^\infty \frac{e^{-xu} u^{s-1}}{e^u - 1} \,\mathrm{d}u \\
&= \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-xz} z^{s-1}}{e^z - 1} \,\mathrm{d}z,
\end{aligned}$$

where we have used the substitutions $z = e^{i\pi} u$ along L_1 and $z = e^{-i\pi} u$ along L_2, and have replaced the dummy variable u by z in the final line.

Thus, we have proved that (167) holds as an identity for $\text{Re}(s) > 1$ and $\text{Re}(x) > -1$. Also, the right-hand side of (167) is analytic for $\text{Re}(x) > -1$ and all $s \in \mathbb{C}\backslash\{1\}$, since the integrand is finite along the contour and entire in x and s. □

Lemma 3.1.4. *If $s = \sigma + it$ is a complex variable with $\sigma, t \in \mathbb{R}$ and $\sigma > 0$, and x is a real variable with $0 < x < \infty$, then the modified Hurwitz zeta function $\zeta_1(x, s)$ can be expressed as*

$$\begin{aligned}
\zeta_1(x, s) = \frac{\chi(s)}{(2\pi)^s} \Bigg(&\int_0^\alpha \frac{e^{i(1+x)u} - e^{-ixu}}{1 - e^{iu}} u^{s-1} \,\mathrm{d}u \\
&+ e^{i\pi s/2} \int_{-i\alpha}^{\infty e^{i\phi_2}} \frac{e^{-(1+x)z}}{1 - e^{-z}} z^{s-1} \,\mathrm{d}z + e^{-i\pi s/2} \int_{i\alpha}^{\infty e^{i\phi_1}} \frac{e^{-(1+x)z}}{1 - e^{-z}} z^{s-1} \,\mathrm{d}z \Bigg), \quad (169)
\end{aligned}$$

for any given α, ϕ_1, ϕ_2 with $0 < \alpha < 2\pi$ and $-\frac{\pi}{2} < \phi_1, \phi_2 < \frac{\pi}{2}$.

Proof. We start with the expression (167) for $\zeta_1(x, s)$, and split the Hankel contour into the three parts L_1, L_2, L_3 defined in Definition 3.1.2.

Firstly, by using Cauchy's theorem and then substituting $z = e^{i\pi} u, z = e^{-i\pi} u$ respectively, the integrals along L_1 and L_2 become:

$$\begin{aligned}
\int_{L_1} \frac{e^{xz} z^{s-1}}{e^{-z} - 1} \,\mathrm{d}z &= \int_{i\alpha}^{\infty e^{i\pi}} \frac{e^{(1+x)z} z^{s-1}}{1 - e^z} \,\mathrm{d}z = e^{i\pi s} \int_{-i\alpha}^{\infty} \frac{e^{-(1+x)u}}{1 - e^{-u}} u^{s-1} \,\mathrm{d}u; \\
\int_{L_2} \frac{e^{xz} z^{s-1}}{e^{-z} - 1} \,\mathrm{d}z &= -\int_{-i\alpha}^{\infty e^{-i\pi}} \frac{e^{(1+x)z} z^{s-1}}{1 - e^z} \,\mathrm{d}z = -e^{-i\pi s} \int_{i\alpha}^{\infty} \frac{e^{-(1+x)u}}{1 - e^{-u}} u^{s-1} \,\mathrm{d}u.
\end{aligned}$$

For the integral along L_3, we split the integrand as follows:

$$\int_{L_3} \frac{e^{xz} z^{s-1}}{e^{-z}-1}\,\mathrm{d}z = -\int_{L_3} \frac{e^{-(1+x)z}}{1-e^{-z}} z^{s-1}\,\mathrm{d}z + \int_{L_3} \frac{e^{-(1+x)z}-e^{xz}}{1-e^{-z}} z^{s-1}\,\mathrm{d}z.$$

For the first of these two integrals, the integrand decays as $z \to +\infty$, and so Cauchy's theorem tells us that $\int_{L_3} = \int_{-i\alpha}^{\infty} - \int_{i\alpha}^{\infty}$. For the second one, the integrand behaves like $-(1+2x)z^{s-1}$ for z near 0, so the integral is finite even around $z=0$ (since we have assumed $\mathrm{Re}(s) > 0$). This means the contour of integration can be deformed to the straight line-segment from $-i\alpha$ to $i\alpha$, and the integral can be simplified as follows:

$$\begin{aligned}
&\int_{-i\alpha}^{0} \frac{e^{-(1+x)z}-e^{xz}}{1-e^{-z}} z^{s-1}\,\mathrm{d}z + \int_{0}^{i\alpha} \frac{e^{-(1+x)z}-e^{xz}}{1-e^{-z}} z^{s-1}\,\mathrm{d}z \\
&= -e^{-i\pi s/2} \int_0^{\alpha} \frac{e^{i(1+x)u}-e^{-ixu}}{1-e^{iu}} u^{s-1}\,\mathrm{d}u + e^{i\pi s/2} \int_0^{\alpha} \frac{e^{-i(1+x)u}-e^{ixu}}{1-e^{-iu}} u^{s-1}\,\mathrm{d}u \\
&= -e^{-i\pi s/2} \int_0^{\alpha} \frac{e^{i(1+x)u}-e^{-ixu}}{1-e^{iu}} u^{s-1}\,\mathrm{d}u + e^{i\pi s/2} \int_0^{\alpha} \frac{e^{-ixu}-e^{i(1+x)u}}{e^{iu}-1} u^{s-1}\,\mathrm{d}u \\
&= 2i\sin\left(\tfrac{\pi s}{2}\right) \int_0^{\alpha} \frac{e^{i(1+x)u}-e^{-ixu}}{1-e^{iu}} u^{s-1}\,\mathrm{d}u.
\end{aligned}$$

Summing up the expressions derived for the integrals along L_1, L_2, and L_3, we find that (167) yields:

$$\begin{aligned}
\zeta_1(x,s) = \frac{\Gamma(1-s)}{2\pi i}\Bigg(& e^{i\pi s}\int_{-i\alpha}^{\infty} \frac{e^{-(1+x)u}}{1-e^{-u}} u^{s-1}\,\mathrm{d}u - e^{-i\pi s}\int_{i\alpha}^{\infty} \frac{e^{-(1+x)u}}{1-e^{-u}} u^{s-1}\,\mathrm{d}u \\
& + \int_{i\alpha}^{\infty} \frac{e^{-(1+x)z}}{1-e^{-z}} z^{s-1}\,\mathrm{d}z - \int_{-i\alpha}^{\infty} \frac{e^{-(1+x)z}}{1-e^{-z}} z^{s-1}\,\mathrm{d}z \\
& + 2i\sin\left(\tfrac{\pi s}{2}\right)\int_0^{\alpha} \frac{e^{i(1+x)u}-e^{-ixu}}{1-e^{iu}} u^{s-1}\,\mathrm{d}u \Bigg) \\
= \tfrac{\Gamma(1-s)}{\pi}\sin\left(\tfrac{\pi s}{2}\right)\Bigg(& e^{i\pi s/2}\int_{-i\alpha}^{\infty} \frac{e^{-(1+x)z}}{1-e^{-z}} z^{s-1}\,\mathrm{d}z + e^{-i\pi s/2}\int_{i\alpha}^{\infty} \frac{e^{-(1+x)z}}{1-e^{-z}} z^{s-1}\,\mathrm{d}z \\
& + \int_0^{\alpha} \frac{e^{i(1+x)u}-e^{-ixu}}{1-e^{iu}} u^{s-1}\,\mathrm{d}u \Bigg).
\end{aligned}$$

In this final expression, the integrands of the first two integrals decay exponentially as z tends to infinity in the right half plane. Thus, by Cauchy's theorem, the upper limits of these integrals can be replaced by $\infty e^{i\phi_1}$ and $\infty e^{i\phi_2}$ respectively for any $\phi_1, \phi_2 \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$. The expression outside the large parentheses is precisely $\frac{\chi(s)}{(2\pi)^s}$, so the result follows. $\square$

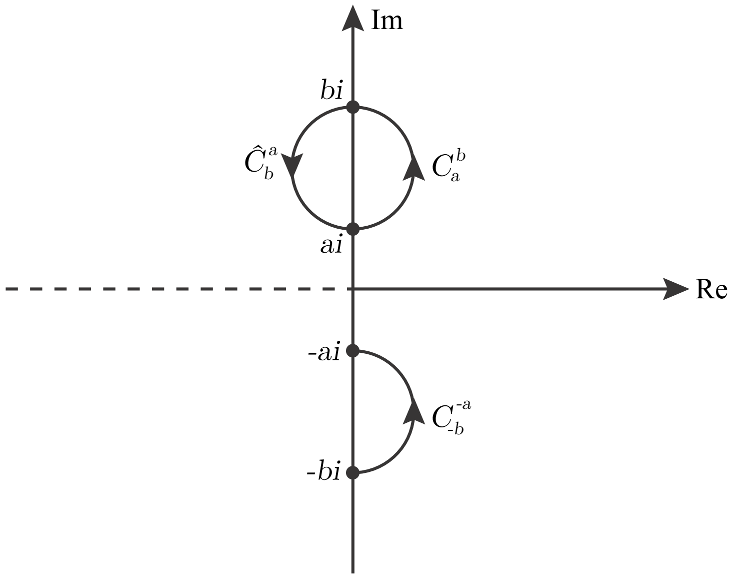

Figure 4: The contours C_a^b, $\hat{C}_b^a$, and C_{-b}^{-a}

Definition 3.1.5. For $a, b \in \mathbb{R}$ with $a < b$, the curves C_a^b and $\hat{C}_b^a$ are defined as follows:

$$C_a^b = \left\{ \frac{i(a+b)}{2} + \frac{b-a}{2} e^{i\theta} \,:\, \theta \in \Big(-\frac{\pi}{2}, \frac{\pi}{2} \Big) \right\};$$
$$\hat{C}_b^a = \left\{ \frac{i(a+b)}{2} + \frac{b-a}{2} e^{i\theta} \,:\, \theta \in \Big(-\pi, -\frac{\pi}{2} \Big) \cup \Big(\frac{\pi}{2}, \pi \Big) \right\}.$$

In other words, C_a^b is the semicircular contour from ia to ib passing upwards through the right half plane, while $\hat{C}_b^a$ is the semicircular contour from ib to ia passing downwards through the left half plane. The two together form a full circular contour, as shown in Figure 4.

The following theorem establishes our main exact formula (166) for the Hurwitz zeta function.

Theorem 3.1.6. *If $s = \sigma + it$ is a complex variable with $\sigma, t \in \mathbb{R}, 0 < \sigma \leq 1, 0 < t < \infty$, and x is a real variable with $0 < x < \infty$, then the modified Hurwitz zeta function $\zeta_1(x, s)$ can be expressed as*

$$\zeta_1(x,s) = \chi(s) \left(\sum_{m=1}^{\lfloor \eta/2\pi \rfloor} e^{-2\pi i m x} m^{s-1} - G_B(\sigma,t;\eta;x) + G_L(\sigma,t;\eta;x) + G_U(\sigma,t;\eta;x) \right), \quad (170)$$

for any given η, ϕ_1, ϕ_2 *with* $0 < \eta < \infty$ *and* $-\frac{\pi}{2} < \phi_1, \phi_2 < \frac{\pi}{2}$, *where*

$$G_B(\sigma, t; \eta; x) := \tfrac{e^{-i\pi s/2}}{(2\pi)^s} \int_{\hat{C}_\eta^0} \frac{e^{(1+x)z} - e^{-xz}}{1 - e^z} z^{s-1}\, \mathrm{d}z, \tag{171}$$

$$G_L(\sigma, t; \eta; x) := \tfrac{e^{i\pi s/2}}{(2\pi)^s} \int_{-i\eta}^{\infty e^{i\phi_2}} \frac{e^{-(1+x)z}}{1 - e^{-z}} z^{s-1}\, \mathrm{d}z, \tag{172}$$

$$G_U(\sigma, t; \eta; x) := \tfrac{e^{-i\pi s/2}}{(2\pi)^s} \int_{i\eta}^{\infty e^{i\phi_1}} \frac{e^{-(1+x)z}}{1 - e^{-z}} z^{s-1}\, \mathrm{d}z. \tag{173}$$

Proof. We start with the result of Lemma 3.1.4. By Cauchy's theorem, the contours of integration in the second and third integrals in (169) can be deformed so as to run first from $\pm i\alpha$ to $\pm i\eta$ and then out to infinity:

$$\int_{-i\alpha}^{\infty e^{i\phi_2}} = -\int_{C_{-\eta}^{-\alpha}} + \int_{-i\eta}^{\infty e^{i\phi_2}} \quad ; \quad \int_{i\alpha}^{\infty e^{i\phi_1}} = \int_{C_\alpha^\eta} + \int_{i\eta}^{\infty e^{i\phi_1}}.$$

Hence, the sum of the last two terms in (169) is equal to the sum of $(2\pi)^s G_L(\sigma, t; \eta; x)$ and $(2\pi)^s G_U(\sigma, t; \eta; x)$ with the following expression:

$$-\, e^{i\pi s/2} \int_{C_{-\eta}^{-\alpha}} \frac{e^{-(1+x)z}}{1 - e^{-z}} z^{s-1}\, \mathrm{d}z + e^{-i\pi s/2} \int_{C_\alpha^\eta} \frac{e^{-(1+x)z}}{1 - e^{-z}} z^{s-1}\, \mathrm{d}z. \tag{174}$$

The first term of this expression, after the substitution $z = ue^{-i\pi}$, becomes

$$\begin{aligned} &-\, e^{-i\pi s/2} \int_{\hat{C}_\eta^\alpha} \frac{e^{(1+x)u}}{1 - e^u} u^{s-1}\, \mathrm{d}u \\ &= e^{-i\pi s/2} \int_{\hat{C}_\eta^\alpha} \frac{e^{xu}}{1 - e^{-u}} u^{s-1}\, \mathrm{d}u \\ &= e^{-i\pi s/2} \int_{\hat{C}_\eta^\alpha} \frac{e^{-(1+x)z}}{1 - e^{-z}} z^{s-1}\, \mathrm{d}z + e^{-i\pi s/2} \int_{\hat{C}_\eta^\alpha} \frac{e^{xz} - e^{-(1+x)z}}{1 - e^{-z}} z^{s-1}\, \mathrm{d}z. \end{aligned}$$

So the expression (174) can be rewritten as:

$$e^{-i\pi s/2} \int_{C'} \frac{e^{-(1+x)z}}{1 - e^{-z}} z^{s-1}\, \mathrm{d}z + e^{-i\pi s/2} \int_{\hat{C}_\eta^\alpha} \frac{e^{xz} - e^{-(1+x)z}}{1 - e^{-z}} z^{s-1}\, \mathrm{d}z,$$

where C' is the circle with centre $\frac{\alpha+\eta}{2}$ formed by combining the two semicircles C_α^η and $\hat{C}_\eta^\alpha$, as seen in Figure 4. The residue theorem lets us compute this circular integral explicitly:

$$\int_{C'} \frac{e^{-(1+x)z}}{1 - e^{-z}} z^{s-1}\, \mathrm{d}z = \sum_{m=1}^{\lfloor \eta/2\pi \rfloor} 2\pi i \mathrm{Res}_{2\pi m i}\left(\frac{e^{-(1+x)z}}{1 - e^{-z}} z^{s-1} \right) = \sum_{m=1}^{\lfloor \eta/2\pi \rfloor} e^{-2\pi i m x} m^{s-1} (2\pi i)^s.$$

So the expression (174) can be rewritten as

$$(2\pi)^s \sum_{m=1}^{\lfloor \eta/2\pi \rfloor} e^{-2\pi i m x} m^{s-1} + e^{-i\pi s/2} \int_{\hat{C}_\eta^\alpha} \frac{e^{xz} - e^{-(1+x)z}}{1 - e^{-z}} z^{s-1} \, \mathrm{d}z,$$

or equivalently (by Cauchy's theorem) as

$$\begin{aligned}(2\pi)^s &\sum_{m=1}^{\lfloor \eta/2\pi \rfloor} e^{-2\pi i m x} m^{s-1} \\ &\qquad + e^{-i\pi s/2} \int_{\hat{C}_\eta^0} \frac{e^{xz} - e^{-(1+x)z}}{1 - e^{-z}} z^{s-1} \, \mathrm{d}z + e^{-i\pi s/2} \int_0^{i\alpha} \frac{e^{xz} - e^{-(1+x)z}}{1 - e^{-z}} z^{s-1} \, \mathrm{d}z.\end{aligned}$$

The last term of this expression, after substituting $z = iu$, becomes exactly minus the first integral in (169). Hence, starting from the formula (169) for $\zeta_1(x, s)$, we find:

$$\begin{aligned}\zeta_1(x, s) &= \frac{\chi(s)}{(2\pi)^s} \Bigg(\int_0^\alpha \frac{e^{i(1+x)u} - e^{-ixu}}{1 - e^{iu}} u^{s-1} \, \mathrm{d}u \\ &\qquad\qquad + (2\pi)^s G_L(\sigma, t; \eta; x) + (2\pi)^s G_U(\sigma, t; \eta; x) + (174) \Bigg) \\ &= \frac{\chi(s)}{(2\pi)^s} \Bigg((2\pi)^s G_L(\sigma, t; \eta; x) + (2\pi)^s G_U(\sigma, t; \eta; x) + (2\pi)^s \sum_{m=1}^{\lfloor \eta/2\pi \rfloor} e^{-2\pi i m x} m^{s-1} \\ &\qquad\qquad + e^{-i\pi s/2} \int_{\hat{C}_\eta^0} \frac{e^{xz} - e^{-(1+x)z}}{1 - e^{-z}} z^{s-1} \, \mathrm{d}z \Bigg) \\ &= \chi(s) \Bigg(\sum_{m=1}^{\lfloor \eta/2\pi \rfloor} e^{-2\pi i m x} m^{s-1} + G_L(\sigma, t; \eta; x) + G_U(\sigma, t; \eta; x) - G_B(\sigma, t; \eta; x) \Bigg),\end{aligned}$$

as required. □

Remark 3.1.7. For the particular case of $x = 0$, we find

$$G_B(\sigma, t; \eta; 0) = \tfrac{e^{-i\pi s/2}}{(2\pi)^s} \int_{\hat{C}_\eta^0} \big(- z^{s-1} \big) \, \mathrm{d}z = \frac{1}{(2\pi)^s} \Big(- \frac{\eta^s}{s} \Big),$$

and therefore the identity (170) reduces, as expected, to the formula (20) proved in [57].

The following lemma will be extremely important in the later analysis, since the function D_N will appear a lot as a result of applying an integration by parts approach to the integrals from G_L, G_U, and G_B.

Lemma 3.1.8. *The function D_N defined by*

$$D_N(z; \xi; \sigma, t) \coloneqq \big(\tfrac{\mathrm{d}}{\mathrm{d}z} \cdot \tfrac{1}{\xi - \frac{it}{z}} \big)^N \big(z^{\sigma - 1} \big) \tag{175}$$

can be expressed in the following form for any $N \geq 0$:

$$D_N = \sum_{b=0}^{N} \sum_{c=0}^{N} A_{bc}^{(N)} \left(\frac{t^b \xi^{N-b} \sigma^c z^{N-b}}{(\xi z - it)^{2N}} \right) z^{\sigma - 1}, \tag{176}$$

where $A_{bc}^{(N)}$ is a Gaussian integer with absolute value $\leq (2N-1)!! \coloneqq (1)(3)(5)\dots(2N-3)(2N-1)$ for each b, c.

Proof. Following the argument of [57], we proceed by induction on N.

In the base case $N = 0$, we must also have $b = c = 0$ and so the expression (176) reduces to

$$D_N = A_{00}^{(0)} \left(\frac{t^0 \xi^0 \sigma^0 z^0}{(\xi z - it)^0} \right) z^{\sigma-1} = A_{00}^{(0)} z^{\sigma-1}.$$

By (175), this is valid with $A_{00}^{(0)} = 1 = (-1)!!$. (It makes sense to define $(-1)!! = 1$ in the same way as we ordinarily define $0! = 1$, because $(2N+1)!! = (2N-1)!!(2N+1)$ for all N and $1!! = 1$.)

Now assume that D_N can be written in the form (176) for some fixed $N \geq 0$, and consider D_{N+1}. Using the definition (175), we have:

$$\begin{aligned}
D_{N+1} &= \frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{D_N}{\xi - \frac{it}{z}} \right) = \frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{z D_N}{\xi z - it} \right) \\
&= \frac{\mathrm{d}}{\mathrm{d}z} \left(\sum_{b=0}^{N} \sum_{c=0}^{N} A_{bc}^{(N)} \left(\frac{t^b \xi^{N-b} \sigma^c z^{N+1-b}}{(\xi z - it)^{2N+1}} \right) z^{\sigma-1} \right) \\
&= \sum_{b=0}^{N} \sum_{c=0}^{N} \frac{A_{bc}^{(N)}}{(\xi z - it)^{2N+2}} \Bigg[\left[t^b \xi^{N-b} \sigma^c (N + \sigma - b) z^{N+\sigma-b-1} \right] (\xi z - it) \\
&\qquad\qquad - \left[t^b \xi^{N-b} \sigma^c z^{N+\sigma-b} \right] (2N+1)\xi \Bigg] \\
&= \sum_{b=0}^{N} \sum_{c=0}^{N} \frac{A_{bc}^{(N)}}{(\xi z - it)^{2N+2}} \Bigg[t^b \xi^{N+1-b} \sigma^c (-N - 1 + \sigma - b) z^{N+\sigma-b} \\
&\qquad\qquad - i t^{b+1} \xi^{N-b} \sigma^c (N + \sigma - b) z^{N+\sigma-b-1} \Bigg] \\
&= \sum_{b=0}^{N} \sum_{c=0}^{N} \Bigg(-(N+1+b) A_{bc}^{(N)} \frac{t^b \xi^{N+1-b} \sigma^c z^{N+1-b}}{(\xi z - it)^{2N+2}} + A_{bc}^{(N)} \frac{t^b \xi^{N+1-b} \sigma^{c+1} z^{N+1-b}}{(\xi z - it)^{2N+2}} \\
&\qquad\qquad - i(N-b) A_{bc}^{(N)} \frac{t^{b+1} \xi^{N-b} \sigma^c z^{N-b}}{(\xi z - it)^{2N+2}} - i A_{bc}^{(N)} \frac{t^{b+1} \xi^{N-b} \sigma^{c+1} z^{N-b}}{(\xi z - it)^{2N+2}} \Bigg) z^{\sigma-1}.
\end{aligned}$$

Thus, setting the values of $A_{bc}^{(N+1)}$ as suggested by this expression, we obtain a formula for D_{N+1} in the form of (176). □

Assumption 3.1.9. We shall fix $\epsilon > 0$ and assume that the variable η is never, for any integer n, within a factor of $1 \pm \epsilon$ of the quantity $\frac{t}{x+n}$. In other words, we assume that

$$\forall n \in \mathbb{Z}, \quad \text{either} \quad \eta > (1+\epsilon)\tfrac{t}{x+n} \quad \text{or} \quad \eta < (1-\epsilon)\tfrac{t}{x+n}.$$

The above assumption can be rewritten as

$$\operatorname{dist}\Big(x - \frac{t}{\eta}, \mathbb{Z}\Big) > \frac{\epsilon t}{\eta},$$

or equivalently as

$$\forall n \in \mathbb{Z}, \ \big|(x+n)\eta - t\big| > \epsilon t. \tag{177}$$

Note that to find out whether a given η satisfies (177), it suffices to check for the particular value of n such that $|(x+n)\eta - t|$ is minimal, i.e. for $n = \lfloor \frac{t}{\eta} - x + \frac{1}{2} \rfloor$, the closest integer to $\frac{t}{\eta} - x$. This n may be either a positive or negative integer, depending on the values of x, t, and η.

3.1.3 Asymptotics for G_L

The series $\sum_{n=0}^{\infty} e^{-nz} = \frac{1}{1-e^{-z}}$ is locally uniformly convergent for $\mathrm{Re}(z) > 0$, so we can interchange the series and integral to obtain

$$G_L(\sigma, t; \eta; x) = \tfrac{e^{i\pi s/2}}{(2\pi)^s} \sum_{n=1}^{\infty} \int_{-i\eta}^{\infty e^{i\phi_2}} e^{-(x+n)z} z^{s-1} \,\mathrm{d}z. \tag{178}$$

Repeatedly integrating by parts in the summand gives

$$\int_{-i\eta}^{\infty e^{i\phi_2}} e^{-(x+n)z} z^{s-1} \,\mathrm{d}z = \sum_{j=0}^{N-1} e^{-(x+n)z + it\log z} \Big(\tfrac{1}{x+n-\frac{it}{z}} \cdot \tfrac{\mathrm{d}}{\mathrm{d}z}\Big)^j \Big(\tfrac{z^{\sigma-1}}{x+n-\frac{it}{z}}\Big)\Bigg|_{z=-i\eta}$$
$$+ \int_{-i\eta}^{\infty e^{i\phi_2}} e^{-(x+n)z + it\log z} D_N(z; x+n; \sigma, t) \,\mathrm{d}z \tag{179}$$

for any $N \in \mathbb{N}$, where D_N is defined by (175).

In what follows, we shall take $\phi_2 = 0$, so that $z \in -i\eta + \mathbb{R}^+$.

Lemma 3.1.10. *D_N can be uniformly estimated in either of the following ways, both valid for* $\mathrm{Im}(z) < 0$ *and* $\xi > 0$*:*

$$D_N(z; \xi; \sigma, t) = O\big((2N-1)!!(N+1)^2 |z|^{\sigma-N-1} \xi^{-N}\big); \tag{180}$$
$$D_N(z; \xi; \sigma, t) = O\big((2N-1)!!(N+1)^2 |z|^{\sigma-1} t^{-N}\big). \tag{181}$$

Proof. By (176), we have the following two expressions for D_N:

$$D_N(z;\xi;\sigma,t) = O\Bigg((2N-1)!!|z|^{\sigma-1} \sum_{b=0}^{N}\sum_{c=0}^{N} \left|\tfrac{t}{\xi z}\right|^{b} |\sigma|^{c} \left|\tfrac{\xi z}{(\xi z - it)^2}\right|^{N} \Bigg);$$
$$D_N(z;\xi;\sigma,t) = O\Bigg((2N-1)!!|z|^{\sigma-1} \sum_{b=0}^{N}\sum_{c=0}^{N} \left|\tfrac{\xi z}{t}\right|^{N-b} |\sigma|^{c} \left|\tfrac{t}{(\xi z - it)^2}\right|^{N} \Bigg).$$

Since $0 < \sigma \leq 1$ and $|\xi z - it|$ is greater than both $|\xi z|$ and t (by our assumption on z and the fact that ξ and t are positive reals), we can simplify these estimates as follows.

Case 1: $|\xi z| > t$.

In this case,

$$\begin{aligned} D_N &= O\Bigg((2N-1)!!|z|^{\sigma-1} \sum_{b=0}^{N} \left|\tfrac{t}{\xi z}\right|^{b} (N+1) \left|\tfrac{\xi z}{(\xi z)^2}\right|^{N} \Bigg) \\ &= O\Big((2N-1)!!(N+1)^2 |z|^{\sigma-N-1} \xi^{-N} \Big). \end{aligned}$$

Case 2: $|\xi z| < t$.

In this case,

$$\begin{aligned} D_N &= O\Bigg((2N-1)!!|z|^{\sigma-1} \sum_{b=0}^{N} \left|\tfrac{\xi z}{t}\right|^{N-b} (N+1) \left|\tfrac{t}{t^2}\right|^{N} \Bigg) \\ &= O\Big((2N-1)!!(N+1)^2 |z|^{\sigma-1} t^{-N} \Big). \end{aligned}$$

In both cases, we have $D_N = O\Big((2N-1)!!(N+1)^2|z|^{\sigma-1} \max\big(\xi|z|, t\big)^{-N}\Big)$, from which both the estimates (180) and (181) follow. Note also that in each case the bound is uniform in all variables: in fact, the O-constant can be taken to be 1. $\square$

Lemma 3.1.11. *We have the following estimate for G_L, uniform in σ, t, η, x, and $N \geq 1$:*

$$G_L(\sigma,t;\eta;x) = \frac{e^{i\pi\sigma/2} e^{it\log\eta}}{(2\pi)^s} \sum_{n=1}^{M}\sum_{j=0}^{N-1} e^{i(x+n)\eta} \left[\left(\tfrac{1}{x+n-\frac{it}{z}} \cdot \tfrac{\mathrm{d}}{\mathrm{d}z} \right)^{j} \left(\tfrac{z^{\sigma-1}}{x+n-\frac{it}{z}} \right) \right]_{z=-i\eta} + O\Big((2N-1)!!(N+1)^2 \eta^{\sigma-N-1} \Big),$$

where M is a finite number depending only on N and η, or $M = \infty$ if $\eta \in 2\pi\mathbb{Z}$.

Proof. Employing (178) and (179) (where we have set $\phi_2 = 0$), it suffices to estimate

$$\sum_{n=1}^{\infty} \int_{-i\eta}^{\infty} e^{-(x+n)z + it\log z} D_N(z; x+n; \sigma, t)\,\mathrm{d}z,$$

which can be achieved using Lemma 3.1.10. By equation (180), this expression is given by:

$$
\begin{aligned}
&O\Bigg(\sum_{n=1}^{\infty}\int_{-i\eta}^{\infty} e^{-(x+n)z+it\log z}(2N-1)!!(N+1)^2|z|^{\sigma-N-1}(x+n)^{-N}\,\mathrm{d}z\Bigg)\\
&=O\Bigg((2N-1)!!(N+1)^2\sum_{n=1}^{\infty}\int_{-i\eta}^{\infty} e^{-(x+n)\mathrm{Re}(z)}e^{\pi t/2}\eta^{\sigma-N-1}(x+n)^{-N}\,\mathrm{d}z\Bigg)\\
&=O\Bigg((2N-1)!!(N+1)^2e^{\pi t/2}\eta^{\sigma-N-1}\sum_{n=1}^{\infty}(x+n)^{-N}\int_{0}^{\infty} e^{-(x+n)u}\,\mathrm{d}z\Bigg)\\
&=O\Bigg((2N-1)!!(N+1)^2e^{\pi t/2}\eta^{\sigma-N-1}\sum_{n=1}^{\infty}(x+n)^{-N-1}\Bigg)\\
&=O\Big((2N-1)!!(N+1)^2e^{\pi t/2}\eta^{\sigma-N-1}\Big).
\end{aligned}
$$

Since $e^{i\pi s/2}=e^{i\pi\sigma/2}e^{-\pi t/2}$ and $e^{it\log(-i\eta)}=e^{it\log\eta}e^{\pi t/2}$, using the above estimate together with (178) and (179) yields:

$$
\begin{aligned}
G_L(\sigma,t;\eta;x)&=\tfrac{e^{i\pi\sigma/2}e^{-\pi t/2}}{(2\pi)^s}\sum_{n=1}^{\infty}\int_{-i\eta}^{\infty e^{i\phi_2}} e^{-(x+n)z}z^{s-1}\,\mathrm{d}z\\
&=\frac{e^{i\pi\sigma/2}e^{it\log\eta}}{(2\pi)^s}\sum_{n=1}^{\infty}\sum_{j=0}^{N-1}e^{i(x+n)\eta}\left[\left(\tfrac{1}{x+n-\frac{it}{z}}\cdot\tfrac{\mathrm{d}}{\mathrm{d}z}\right)^j\left(\tfrac{z^{\sigma-1}}{x+n-\frac{it}{z}}\right)\right]_{z=-i\eta}\\
&\qquad+\frac{e^{i\pi\sigma/2}e^{-\pi t/2}}{(2\pi)^s}\sum_{n=1}^{\infty}\int_{-i\eta}^{\infty e^{i\phi_2}} e^{-(x+n)z+it\log z}D_N(z;x+n;\sigma,t)\,\mathrm{d}z\\
&=\frac{e^{i\pi\sigma/2}e^{it\log\eta}}{(2\pi)^s}\sum_{n=1}^{\infty}\sum_{j=0}^{N-1}e^{i(x+n)\eta}\left[\left(\tfrac{1}{x+n-\frac{it}{z}}\cdot\tfrac{\mathrm{d}}{\mathrm{d}z}\right)^j\left(\tfrac{z^{\sigma-1}}{x+n-\frac{it}{z}}\right)\right]_{z=-i\eta}\\
&\qquad+O\Big((2N-1)!!(N+1)^2\eta^{\sigma-N-1}\Big).
\end{aligned}
$$

The uniformity of the O-bound is inherited from Lemma 3.1.10. In order to derive the final result, we just need to find $M(N,\eta)$ large enough so that

$$
\sum_{n=M+1}^{\infty}\sum_{j=0}^{N-1}e^{i(x+n)\eta}\left[\left(\tfrac{1}{x+n-\frac{it}{z}}\cdot\tfrac{\mathrm{d}}{\mathrm{d}z}\right)^j\left(\tfrac{z^{\sigma-1}}{x+n-\frac{it}{z}}\right)\right]_{z=-i\eta}=O\Big((2N-1)!!(N+1)^2\eta^{\sigma-N-1}\Big). \quad (182)
$$

Using the definition of D_N and the bound (180) for D_N, we can estimate the left hand side of (182) as follows:

$$
\begin{aligned}
&\sum_{n=M+1}^{\infty}\sum_{j=0}^{N-1}\frac{e^{i(x+n)\eta}}{x+n+\frac{t}{\eta}}D_j(-i\eta;x+n;\sigma,t)\\
&=\sum_{n=M+1}^{\infty}\frac{e^{i(x+n)\eta}(-i\eta)^{\sigma-1}}{x+n+\frac{t}{\eta}}+\sum_{n=M+1}^{\infty}\sum_{j=1}^{N-1}O\bigg(\frac{1}{x+n+\frac{t}{\eta}}(2j-1)!!(j+1)^2\eta^{\sigma-j-1}(x+n)^{-j}\bigg)\\
&=O\bigg(\eta^{\sigma-1}\sum_{n=M+1}^{\infty}\frac{e^{in\eta}}{x+n+\frac{t}{\eta}}\bigg)+\sum_{j=1}^{N-1}O\bigg((2j-1)!!(j+1)^2\eta^{\sigma-j-1}\sum_{n=M+1}^{\infty}(x+n)^{-j-1}\bigg)\\
&=O\bigg(\eta^{\sigma-1}\sum_{n=M+1}^{\infty}\frac{e^{in\eta}}{x+n}\bigg)+O\bigg((2N-3)!!N^2\sum_{j=1}^{N-1}\eta^{\sigma-j-1}\sum_{n=M+1}^{\infty}(x+n)^{-j-1}\bigg).
\end{aligned}
$$

All of the infinite series in this expression are convergent, so we can simply choose M large enough so that

$$\sum_{n=M+1}^{\infty}\frac{e^{in\eta}}{x+n}\leq\eta^{-N}$$

and

$$\sum_{n=M+1}^{\infty}(x+n)^{-j-1}\leq\eta^{j-N}$$

for $j=1,2,\ldots,N-1$. In the second of these inequalities, the left hand side is decreasing in j while the right hand side is increasing in j, so we can simplify the conditions to

$$\sum_{n=M+1}^{\infty}e^{in\eta}n^{-1}\leq\eta^{-N} \tag{183}$$

and

$$\sum_{n=M+1}^{\infty}n^{-2}\leq\eta^{1-N}. \tag{184}$$

For any M satisfying (183) and (184), we have the required bound (182), and so the final result holds. Note that if $\eta\in 2\pi\mathbb{Z}$ then the series in (183) is divergent and so we need $M=\infty$. □

3.1.4 Asymptotics for G_U

As before, the series $\sum_{n=0}^{\infty}e^{-nz}=\frac{1}{1-e^{-z}}$ is locally uniformly convergent for $\mathrm{Re}(z)>0$, so we can interchange the series and integral to obtain

$$G_U(\sigma,t;\eta;x)=\frac{e^{-i\pi s/2}}{(2\pi)^s}\sum_{n=1}^{\infty}\int_{i\eta}^{\infty e^{i\phi_1}}e^{-(x+n)z}z^{s-1}\,\mathrm{d}z. \tag{185}$$

Let us fix $\phi_1 = \frac{\pi}{2}$, so that $z \in i[\eta, \infty)$. Now the integrand is $e^{-(x+n)z+it} z^{\sigma-1}$, which has a stationary point iff $-(x+n) + \frac{it}{z} = 0$, i.e. at $z = \frac{it}{x+n}$. So there is a stationary point in the interval of integration iff

$$\tfrac{it}{x+n} \in i[\eta, \infty), \text{ i.e. } \eta \le \tfrac{t}{x+n} < \infty, \text{ i.e. } n \le \tfrac{t}{\eta} - x.$$

This is the first place we need to use Assumption 3.1.9. The inequality (177) can be rearranged in terms of n, since its opposite statement rearranges as follows:

$$\begin{aligned} |(x+n)\eta - t| \ge \epsilon t &\Leftrightarrow (1-\epsilon)t \le (x+n)\eta \le (1+\epsilon)t \\ &\Leftrightarrow (1-\epsilon)\tfrac{t}{\eta} - x \le n \le (1+\epsilon)\tfrac{t}{\eta} - x. \end{aligned}$$

So we need to consider two separate cases, namely $n < (1-\epsilon)\frac{t}{\eta} - x$ and $n > (1+\epsilon)\frac{t}{\eta} - x$. In other words, the sum over n appearing in (185) needs to be split into two separate subseries. When $n > (1+\epsilon)\frac{t}{\eta} - x$, there is no stationary point in the interval of integration and we can use integration by parts as before. When $n < (1-\epsilon)\frac{t}{\eta} - x$, we shall rewrite the integral along $i[\eta, \infty)$ as the difference of an integral along $i[0, \eta)$, which no longer contains a stationary point, and an integral along $i[0, \infty)$, which can be computed explicitly.

In analogy with equation (179), repeatedly integrating by parts in the summand of (185) gives

$$\begin{aligned} \int_{i\eta}^{i\infty} e^{-(x+n)z} z^{s-1}\,\mathrm{d}z = \sum_{j=0}^{N-1} e^{-(x+n)z+it\log z} \left(\tfrac{1}{x+n-\frac{it}{z}} \cdot \tfrac{\mathrm{d}}{\mathrm{d}z}\right)^j \left(\tfrac{z^{\sigma-1}}{x+n-\frac{it}{z}}\right)\Bigg|_{z=i\eta} \\ + \int_{i\eta}^{i\infty} e^{-(x+n)z+it\log z} D_N(z; x+n; \sigma, t)\,\mathrm{d}z. \quad (186) \end{aligned}$$

Similarly,

$$\begin{aligned} \int_{0}^{i\eta} e^{-(x+n)z} z^{s-1}\,\mathrm{d}z = -\sum_{j=0}^{N-1} e^{-(x+n)z+it\log z} \left(\tfrac{1}{x+n-\frac{it}{z}} \cdot \tfrac{\mathrm{d}}{\mathrm{d}z}\right)^j \left(\tfrac{z^{\sigma-1}}{x+n-\frac{it}{z}}\right)\Bigg|_{z=i\eta} \\ + \int_{0}^{i\eta} e^{-(x+n)z+it\log z} D_N(z; x+n; \sigma, t)\,\mathrm{d}z, \quad (187) \end{aligned}$$

each of (186) and (187) being valid for any $N \in \mathbb{N}$. To derive each of these identities, we have used the fact that for every j, the summand in the $\sum_j$ series tends to zero as $|z|$ tends to either 0 or ∞ with z on the positive imaginary axis. This follows by approximating each part of the summand, e.g. by a power of z.

Lemma 3.1.12. *If $n < (1-\epsilon)\frac{t}{\eta} - x$ and $z \in i[0, \eta]$, then*

$$D_N(z; x+n; \sigma, t) = O\Big((2N-1)!!(N+1)^2 |z|^{\sigma-1} t^{-N} \epsilon^{-2N}\Big). \quad (188)$$

If $n > (1+\epsilon)\frac{t}{\eta} - x$ *and* $z \in i[\eta, \infty)$*, then*

$$D_N(z; x+n; \sigma, t) = O\Big((2N-1)!!(N+1)^2 |z|^{\sigma-N-1}(x+n)^{-N}\epsilon^{-2N}(1+\epsilon)^{2N}\Big). \quad (189)$$

Both of these estimates are uniform in all parameters.

Proof. The argument here is similar to the argument used in Lemma 3.1.10, starting from the expression (176) for D_N.

Case 1: $\boldsymbol{n < (1-\epsilon)\frac{t}{\eta} - x, z \in i[0, \eta]}$.

In this case, $|(x+n)z| \le (x+n)\eta < (1-\epsilon)t$, and thus

$$\begin{aligned} D_N(z; x+n; \sigma, t) &= O\left((2N-1)!! |z|^{\sigma-1} \sum_{b=0}^{N} \sum_{c=0}^{N} \left|\tfrac{(x+n)z}{t}\right|^{N-b} |\sigma|^c \left|\tfrac{t}{((x+n)z-it)^2}\right|^N \right) \\ &= O\Big((2N-1)!!(N+1)^2 |z|^{\sigma-1} \left|\tfrac{t}{((x+n)z-it)^2}\right|^N \Big). \end{aligned}$$

By assumption, $(x+n)z$ is positive imaginary with modulus at most $(1-\epsilon)t$, and so $\big|(x+n)z - it\big| > \epsilon t$. Therefore

$$D_N = O\Big((2N-1)!!(N+1)^2|z|^{\sigma-1}t^N(\epsilon t)^{-2N}\Big),$$

as required.

Case 2: $\boldsymbol{n > (1+\epsilon)\frac{t}{\eta} - x, z \in i[\eta, \infty)}$.

In this case, $|(x+n)z| \ge (x+n)\eta > (1+\epsilon)t$, and thus

$$\begin{aligned} D_N(z; x+n; \sigma, t) &= O\left((2N-1)!! |z|^{\sigma-1} \sum_{b=0}^{N} \sum_{c=0}^{N} \left|\tfrac{t}{(x+n)z}\right|^{b} |\sigma|^c \left|\tfrac{(x+n)z}{((x+n)z-it)^2}\right|^N \right) \\ &= O\Big((2N-1)!!(N+1)^2 |z|^{\sigma-1} \left|\tfrac{(x+n)z}{((x+n)z-it)^2}\right|^N \Big). \end{aligned}$$

By assumption, $(x+n)z$ is positive imaginary with modulus at least $(1+\epsilon)t$, and so $\big|(x+n)z - it\big| > \frac{\epsilon}{1+\epsilon}\big|(x+n)z\big|$. Therefore

$$D_N = O\Big((2N-1)!!(N+1)^2|z|^{\sigma-1}\big|(x+n)z\big|^{-N}\big(\tfrac{\epsilon}{1+\epsilon}\big)^{-2N}\Big),$$

which yields the desired estimate.

As in Lemma 3.1.10, all bounds are uniform and the O-constants can each be taken to be 1. ☐

Lemma 3.1.13. *We have the following two estimates, uniform in* σ*,* t*,* η*,* x*,* ϵ*, and*

$N \geq 1$ *satisfying Assumption 3.1.9:*

$$\sum_{n=1}^{\left\lfloor \frac{t}{\eta}-x \right\rfloor} \int_0^{i\eta} e^{-(x+n)z+it\log z} D_N(z;x+n;\sigma,t)\,\mathrm{d}z$$
$$= O\big((2N+1)!!(N+1)^2 e^{-\pi t/2}\sigma^{-1}\eta^{\sigma-N-1}\epsilon^{-2N-2}\big),$$

and

$$\sum_{n=\left\lceil \frac{t}{\eta}-x \right\rceil}^{\infty} \int_{i\eta}^{i\infty} e^{-(x+n)z+it\log z} D_N(z;x+n;\sigma,t)\,\mathrm{d}z$$
$$= O\big((2N+1)!!(N+1)^2 e^{-\pi t/2}\eta^{\sigma-N-1}\epsilon^{-2N-2}(1+\epsilon)^{2N+2}\big).$$

Proof. We shall use the estimates from Lemma 3.1.12. Let ${}_UI_1$ and ${}_UI_2$ denote the two expressions we need to estimate.

First, by (188) we find the following estimate:

$$\begin{aligned}
{}_UI_1 &= O\left(\sum_{n=1}^{\left\lfloor \frac{t}{\eta}-x \right\rfloor} \int_0^{i\eta} e^{-(x+n)z+it\log z}(2N-1)!!(N+1)^2|z|^{\sigma-1}t^{-N}\epsilon^{-2N}\,\mathrm{d}z\right)\\
&= O\left((2N-1)!!(N+1)^2 t^{-N}\epsilon^{-2N}\sum_{n=1}^{\left\lfloor \frac{t}{\eta}-x \right\rfloor}\int_0^{i\eta} e^{-\pi t/2}|z|^{\sigma-1}\,\mathrm{d}z\right)\\
&= O\left((2N-1)!!(N+1)^2 e^{-\pi t/2}\left(\tfrac{\eta^\sigma}{\sigma}\right)t^{-N}\epsilon^{-2N}\sum_{n=1}^{\left\lfloor \frac{t}{\eta}-x \right\rfloor} 1\right)\\
&= O\big((2N-1)!!(N+1)^2 e^{-\pi t/2}\sigma^{-1}\eta^{\sigma-1}t^{-N+1}\epsilon^{-2N}\big).
\end{aligned}$$

We can assume $\eta < t$ (otherwise the series is non-existent), so $t^{-N+1} < \eta^{-N+1}$ and therefore

$$ {}_UI_1 = O\big((2N-1)!!(N+1)^2 e^{-\pi t/2}\sigma^{-1}\eta^{\sigma-N}\epsilon^{-2N}\big). \tag{190}$$

But here the exponent of η is just one too big. So we apply integration by parts once more to the original expression for ${}_UI_1$:

$$ {}_UI_1 = \sum_{n=1}^{\left\lfloor \frac{t}{\eta}-x \right\rfloor}\left(-\left[e^{-(x+n)z+it\log z}\left(\frac{1}{x+n-\frac{it}{z}}\right)D_N(z;x+n;\sigma,t)\right]_0^{i\eta}\right.$$
$$\left. + \int_0^{i\eta} e^{-(x+n)z+it\log z} D_{N+1}(z;x+n;\sigma,t)\,\mathrm{d}z\right).$$

Using equation (188) again for the first half of this and equation (190) (with N replaced by $N+1$) for the second half, we find:

$$\begin{aligned}
{}_U I_1 &= O\Bigg(\sum_{n=1}^{\left\lfloor \frac{t}{\eta} - x \right\rfloor} \Big[\tfrac{e^{-\pi t/2} z}{(x+n)z - it} (2N-1)!!(N+1)^2 |z|^{\sigma-1} t^{-N} \epsilon^{-2N} \Big]_{z=i\eta} \Bigg) \\
&\qquad\qquad\qquad + O\big((2N+1)!!(N+2)^2 e^{-\pi t/2} \sigma^{-1} \eta^{\sigma-N-1} \epsilon^{-2N-2}\big) \\
&= O\Big(\tfrac{e^{-\pi t/2}\eta}{\epsilon t} (2N+1)!!(N+1) \eta^{\sigma-2} t^{-N+1} \epsilon^{-2N} \Big) \\
&\qquad\qquad\qquad + O\big((2N+1)!!(N+1)^2 e^{-\pi t/2} \sigma^{-1} \eta^{\sigma-N-1} \epsilon^{-2N-2}\big) \\
&= O\Big((2N+1)!! e^{-\pi t/2} \big[(N+1) \eta^{\sigma-1} t^{-N} \epsilon^{-2N-1} + (N+1)^2 \sigma^{-1} \eta^{\sigma-N-1} \epsilon^{-2N-2} \big] \Big) \\
&= O\big((2N+1)!!(N+1)^2 e^{-\pi t/2} \sigma^{-1} \eta^{\sigma-N-1} \epsilon^{-2N-2} \big),
\end{aligned}$$

where we have again used the estimates $(x+n)\eta - t > \epsilon t$ and $t^{-N} < \eta^{-N}$. Once again, all O-constants are uniform.

Second, by (189) we have:

$$\begin{aligned}
{}_U I_2 &= O\Bigg(\sum_{n=\left\lceil \frac{t}{\eta} - x \right\rceil}^{\infty} \int_{i\eta}^{i\infty} e^{-(x+n)z + it\log z} \\
&\qquad\qquad (2N-1)!!(N+1)^2 |z|^{\sigma-N-1} (x+n)^{-N} \epsilon^{-2N} (1+\epsilon)^{2N} \,\mathrm{d}z \Bigg) \\
&= O\Bigg((2N-1)!!(N+1)^2 \epsilon^{-2N} (1+\epsilon)^{2N} \sum_{n=\left\lceil \frac{t}{\eta} - x \right\rceil}^{\infty} \int_{i\eta}^{i\infty} e^{-\pi t/2} |z|^{\sigma-N-1} (x+n)^{-N} \,\mathrm{d}z \Bigg) \\
&= O\Bigg((2N-1)!!(N+1)^2 e^{-\pi t/2} (N-\sigma)^{-1} \eta^{\sigma-N} \epsilon^{-2N} (1+\epsilon)^{2N} \sum_{n=\left\lceil \frac{t}{\eta} - x \right\rceil}^{\infty} (x+n)^{-N} \Bigg) \\
&= O\big((2N-1)!!(N+1)^2 e^{-\pi t/2} \eta^{\sigma-N} \epsilon^{-2N} (1+\epsilon)^{2N} \big), \qquad (191)
\end{aligned}$$

provided that $N \geq 2$.

But here the exponent of η is just one too big. So we apply integration by parts once more to the original expression for ${}_U I_2$:

$$\begin{aligned}
{}_U I_2 = \sum_{n=\left\lceil \frac{t}{\eta} - x \right\rceil}^{\infty} \Bigg(&- \Big[e^{-(x+n)z + it\log z} \Big(\tfrac{1}{x+n-\frac{it}{z}} \Big) D_N(z; x+n; \sigma, t) \Big]_{i\eta}^{i\infty} \\
&+ \int_{i\eta}^{i\infty} e^{-(x+n)z + it\log z} D_{N+1}(z; x+n; \sigma, t) \,\mathrm{d}z \Bigg).
\end{aligned}$$

Using equation (189) again for the first half of this expression and equation (191) (with N replaced by $N+1$, so that our $N \geq 2$ assumption becomes only $N \geq 1$) for the second half, we find:

$$
\begin{aligned}
{}_U I_2 &= O\left(\sum_{n=\left\lceil \frac{t}{\eta} - x \right\rceil}^{\infty} \left[\tfrac{e^{-\pi t/2} z}{(x+n)z - it} (2N-1)!!(N+1)^2 |z|^{\sigma-N-1} (x+n)^{-N} \epsilon^{-2N} (1+\epsilon)^{2N} \right]_{z=i\eta} \right) \\
&\qquad\qquad\qquad + O\big((2N+1)!!(N+2)^2 e^{-\pi t/2} \eta^{\sigma-N-1} \epsilon^{-2N-2} (1+\epsilon)^{2N+2}\big) \\
&= O\left(\sum_{n=\left\lceil \frac{t}{\eta} - x \right\rceil}^{\infty} \tfrac{e^{-\pi t/2}\eta}{\frac{\epsilon}{1+\epsilon}(x+n)\eta} (2N+1)!!(N+1) \eta^{\sigma-N-1} (x+n)^{-N} \epsilon^{-2N} (1+\epsilon)^{2N} \right) \\
&\qquad\qquad\qquad + O\big((2N+1)!!(N+1)^2 e^{-\pi t/2} \eta^{\sigma-N-1} \epsilon^{-2N-2} (1+\epsilon)^{2N+2}\big) \\
&= O\Big(e^{-\pi t/2} (2N+1)!! \Big[(N+1) \eta^{\sigma-N-1} \epsilon^{-2N-1} (1+\epsilon)^{2N+1} \\
&\qquad\qquad\qquad + (N+1)^2 \eta^{\sigma-N-1} \epsilon^{-2N-2} (1+\epsilon)^{2N+2} \Big] \Big) \\
&= O\big((2N+1)!!(N+1)^2 e^{-\pi t/2} \eta^{\sigma-N-1} \epsilon^{-2N-2} (1+\epsilon)^{2N+2}\big),
\end{aligned}
$$

where we have used again the estimate $(x+n)\eta - t > \frac{\epsilon}{1+\epsilon}(x+n)\eta$. Once again, all O-constants are uniform. $\square$

Lemma 3.1.14. *We have the following estimate for G_U, uniform in σ, t, η, x, ϵ, and $N \geq 1$ satisfying Assumption 3.1.9:*

$$
\begin{aligned}
G_U(\sigma, t; \eta; x) &= \frac{e^{-i\pi s/2}}{(2\pi)^s} \sum_{n=1}^{\left\lfloor \frac{t}{\eta} - x \right\rfloor} \frac{\Gamma(s)}{(x+n)^s} \\
&\quad + \frac{e^{-i\pi\sigma/2} e^{it\log\eta}}{(2\pi)^s} \sum_{n=1}^{M} \sum_{j=0}^{N-1} e^{-i(x+n)\eta} \left[\left(\tfrac{1}{x+n-\frac{it}{z}} \cdot \tfrac{\mathrm{d}}{\mathrm{d}z} \right)^j \left(\tfrac{z^{\sigma-1}}{x+n-\frac{it}{z}} \right) \right]_{z=i\eta} \\
&\qquad\qquad + O\big((2N+1)!!(N+1)^2 \sigma^{-1} \eta^{\sigma-N-1} \epsilon^{-2N-2} (1+\epsilon)^{2N+2}\big),
\end{aligned}
$$

where M is a finite number depending only on N and η, or $M = \infty$ if $\eta \in 2\pi\mathbb{Z}$.

Proof. By (185) with $\phi_1 = \frac{\pi}{2}$, we find:

$$\begin{aligned}
G_U(\sigma,t;\eta;x) &= \tfrac{e^{-i\pi s/2}}{(2\pi)^s} \sum_{n=1}^{\infty} \int_{i\eta}^{i\infty} e^{-(x+n)z} z^{s-1}\,\mathrm{d}z \\
&= \frac{e^{-i\pi s/2}}{(2\pi)^s} \Bigg[\sum_{n=1}^{\left\lfloor \frac{t}{\eta}-x \right\rfloor} \left(\int_0^{i\infty} e^{-(x+n)z} z^{s-1}\,\mathrm{d}z - \int_0^{i\eta} e^{-(x+n)z} z^{s-1}\,\mathrm{d}z \right) \\
&\qquad\qquad + \sum_{n=\left\lceil \frac{t}{\eta}-x \right\rceil}^{\infty} \int_{i\eta}^{i\infty} e^{-(x+n)z} z^{s-1}\,\mathrm{d}z \Bigg] \\
&= \frac{e^{-i\pi s/2}}{(2\pi)^s} \Bigg[\sum_{n=1}^{\left\lfloor \frac{t}{\eta}-x \right\rfloor} \left(\frac{\Gamma(s)}{(x+n)^s} - \int_0^{i\eta} e^{-(x+n)z} z^{s-1}\,\mathrm{d}z \right) \\
&\qquad\qquad + \sum_{n=\left\lceil \frac{t}{\eta}-x \right\rceil}^{\infty} \int_{i\eta}^{i\infty} e^{-(x+n)z} z^{s-1}\,\mathrm{d}z \Bigg].
\end{aligned}$$

Substituting (186) and (187) into the above expression yields:

$$\begin{aligned}
G_U = \frac{e^{-i\pi s/2}}{(2\pi)^s} \Bigg[&\sum_{n=1}^{\left\lfloor \frac{t}{\eta}-x \right\rfloor} \frac{\Gamma(s)}{(x+n)^s} - \sum_{n=1}^{\left\lfloor \frac{t}{\eta}-x \right\rfloor} \Bigg(-\sum_{j=0}^{N-1} e^{-(x+n)z+it\log z} \left(\tfrac{1}{x+n-\frac{it}{z}} \cdot \tfrac{\mathrm{d}}{\mathrm{d}z} \right)^j \left(\tfrac{z^{\sigma-1}}{x+n-\frac{it}{z}} \right) \Bigg|_{z=i\eta} \\
&\qquad + \int_0^{i\eta} e^{-(x+n)z+it\log z} D_N(z;x+n;\sigma,t)\,\mathrm{d}z \Bigg) \\
&+ \sum_{n=\left\lceil \frac{t}{\eta}-x \right\rceil}^{\infty} \Bigg(\sum_{j=0}^{N-1} e^{-(x+n)z+it\log z} \left(\tfrac{1}{x+n-\frac{it}{z}} \cdot \tfrac{\mathrm{d}}{\mathrm{d}z} \right)^j \left(\tfrac{z^{\sigma-1}}{x+n-\frac{it}{z}} \right) \Bigg|_{z=i\eta} \\
&\qquad + \int_{i\eta}^{i\infty} e^{-(x+n)z+it\log z} D_N(z;x+n;\sigma,t)\,\mathrm{d}z \Bigg) \Bigg],
\end{aligned}$$

where we have used Jordan's lemma and the fact that $x+n$ is positive to obtain

$$\int_0^{i\infty} e^{-(x+n)z} z^{s-1}\,\mathrm{d}z = (n+x)^{-s}\Gamma(s).$$

Now substituting the results of Lemma 3.1.13 into the expression for G_U yields

$$\begin{aligned}
G_U = \frac{e^{-i\pi s/2}}{(2\pi)^s} \Bigg[&\sum_{n=1}^{\left\lfloor \frac{t}{\eta}-x \right\rfloor} \frac{\Gamma(s)}{(x+n)^s} + \sum_{n=1}^{\infty} \sum_{j=0}^{N-1} e^{-(x+n)z+it\log z} \left(\tfrac{1}{x+n-\frac{it}{z}} \cdot \tfrac{\mathrm{d}}{\mathrm{d}z} \right)^j \left(\tfrac{z^{\sigma-1}}{x+n-\frac{it}{z}} \right) \Bigg|_{z=i\eta} \\
&+ O\big((2N+1)!!(N+1)^2 e^{-\pi t/2} \sigma^{-1} \eta^{\sigma-N-1} \epsilon^{-2N-2} (1+\epsilon)^{2N+2}\big) \Bigg].
\end{aligned}$$

Finally, using $e^{-i\pi s/2} = e^{-i\pi\sigma/2}e^{\pi t/2}$ and $e^{it\log(i\eta)} = e^{it\log\eta}e^{-\pi t/2}$ gives

$$
\begin{aligned}
G_U = \frac{e^{-i\pi s/2}}{(2\pi)^s} \sum_{n=1}^{\left\lfloor \frac{t}{\eta} - x \right\rfloor} \frac{\Gamma(s)}{(x+n)^s} \\
+ \frac{e^{-i\pi\sigma/2} e^{it\log\eta}}{(2\pi)^s} \sum_{n=1}^{\infty} \sum_{j=0}^{N-1} e^{-i(x+n)\eta} \left[\left(\frac{1}{x+n-\frac{it}{z}} \cdot \frac{\mathrm{d}}{\mathrm{d}z} \right)^j \left(\frac{z^{\sigma-1}}{x+n-\frac{it}{z}} \right) \right]_{z=i\eta} \\
+ O\big((2N+1)!!(N+1)^2\sigma^{-1}\eta^{\sigma-N-1}\epsilon^{-2N-2}(1+\epsilon)^{2N+2}\big).
\end{aligned}
$$

The uniformity of the O-bound is inherited from Lemma 3.1.13. In order to derive the final result, we just need to find $M(N,\eta)$ large enough so that

$$
\begin{aligned}
\sum_{n=M+1}^{\infty} \sum_{j=0}^{N-1} e^{-i(x+n)\eta} \left[\left(\frac{1}{x+n-\frac{it}{z}} \cdot \frac{\mathrm{d}}{\mathrm{d}z} \right)^j \left(\frac{z^{\sigma-1}}{x+n-\frac{it}{z}} \right) \right]_{z=i\eta} \\
= O\big((2N+1)!!(N+1)^2\sigma^{-1}\eta^{\sigma-N-1}\epsilon^{-2N-2}(1+\epsilon)^{2N+2}\big). \quad (192)
\end{aligned}
$$

Using the definition of D_N and the bounds (188) and (189) for D_N, we can estimate the left hand side of (192) as follows:

$$
\begin{aligned}
&\sum_{n=M+1}^{\infty} \sum_{j=0}^{N-1} \frac{e^{-i(x+n)\eta}}{x+n-\frac{t}{\eta}} D_j(i\eta; x+n; \sigma, t) \\
&= \sum_{n=M+1}^{\infty} \frac{e^{-i(x+n)\eta}(i\eta)^{\sigma-1}}{x+n-\frac{t}{\eta}} \\
&\quad + \sum_{n=M+1}^{\infty} \sum_{j=1}^{N-1} O\left(\frac{1}{x+n-\frac{t}{\eta}} (2j-1)!!(j+1)^2 \eta^{\sigma-1} \max\big(t, (x+n)\eta\big)^{-j} \epsilon^{-2j}(1+\epsilon)^{2j} \right) \\
&= O\left(\eta^{\sigma-1} \sum_{n=M+1}^{\infty} \frac{e^{in\eta}}{x+n-\frac{t}{\eta}} \right) \\
&\quad + \sum_{j=1}^{N-1} O\left((2j-1)!!(j+1)^2 \eta^{\sigma-1} \epsilon^{-2j}(1+\epsilon)^{2j} \sum_{n=M+1}^{\infty} \frac{1}{x+n-\frac{t}{\eta}} \big((x+n)\eta\big)^{-j} \right).
\end{aligned}
$$

In both Case 1 and Case 2 of Lemma 3.1.12, we have $\big|(x+n)\eta - t\big| > \frac{\epsilon}{1+\epsilon}(x+n)\eta$. So the left hand side of (192) is:

$$
\begin{aligned}
O\left(\eta^{\sigma-1} \sum_{n=M+1}^{\infty} \frac{e^{in\eta}}{\frac{\epsilon}{1+\epsilon}(x+n)} \right) \\
+ O\left((2N-3)!! N^2 \epsilon^{-2N+2}(1+\epsilon)^{2N-2} \sum_{j=1}^{N-1} \eta^{\sigma-j-1} \sum_{n=M+1}^{\infty} \frac{1+\epsilon}{\epsilon}(x+n)^{-j-1} \right).
\end{aligned}
$$

All of the infinite series in this expression are convergent, so we can simply choose M large enough so that

$$\sum_{n=M+1}^{\infty} \frac{e^{in\eta}}{x+n} \leq \eta^{-N}$$

and

$$\sum_{n=M+1}^{\infty} (x+n)^{-j-1} \leq \eta^{j-N}$$

for $j = 1, 2, \ldots, N-1$. These are exactly the same conditions as we found in the proof of Lemma 3.1.11. So for any M satisfying (183) and (184), we have the required bound (192), and the result holds. $\square$

3.1.5 Asymptotics for G_B

The semicircle $\hat{C}_\eta^0$ is in the left half plane, and the series $\sum_{n=0}^{\infty} e^{nz} = \frac{1}{1-e^z}$ is locally uniformly convergent for $\mathrm{Re}(z) < 0$. So we can interchange the series and integral and then use Cauchy's theorem:

$$\begin{aligned} G_B(\sigma, t; \eta; x) &= \tfrac{e^{-i\pi s/2}}{(2\pi)^s} \sum_{n=0}^{\infty} \int_{\hat{C}_\eta^0} \left(e^{(1+x)z} - e^{-xz}\right) e^{nz} z^{s-1} \,\mathrm{d}z \\ &= \tfrac{e^{-i\pi s/2}}{(2\pi)^s} \sum_{n=0}^{\infty} \int_{i\eta}^{0} \left(e^{(n+1+x)z} - e^{(n-x)z}\right) z^{s-1} \,\mathrm{d}z. \end{aligned}$$

Substituting $\tilde{z} = e^{-i\pi} z$ and then changing notation back to z, we find

$$\begin{aligned} G_B &= \tfrac{e^{i\pi s/2}}{(2\pi)^s} \sum_{n=0}^{\infty} \int_{-i\eta}^{0} \left(e^{-(n+1+x)\tilde{z}} - e^{-(n-x)\tilde{z}}\right) \tilde{z}^{s-1} \,\mathrm{d}\tilde{z} \\ &= \tfrac{e^{i\pi s/2}}{(2\pi)^s} \sum_{n=0}^{\infty} \left(\int_{-i\eta}^{0} e^{-(n+1+x)z} z^{s-1} \,\mathrm{d}z - \int_{-i\eta}^{0} e^{-(n-x)z} z^{s-1} \,\mathrm{d}z \right). \end{aligned} \tag{193}$$

The first integrand is $e^{-(n+1+x)z + it\log z} z^{\sigma-1}$, which has a stationary point at $z = \frac{it}{n+1+x}$. This value of z is not in the interval of integration since $\eta > 0$. The second integrand is $e^{-(n-x)z+it\log z} z^{\sigma-1}$, which has a stationary point at $z = \frac{it}{n-x}$. This is within the interval of integration iff

$$\tfrac{it}{n-x} \in -i[0, \eta], \text{ i.e. } 0 \leq \tfrac{t}{x-n} \leq \eta, \text{ i.e. } n \leq x - \tfrac{t}{\eta}.$$

Hence, in analogy with the asymptotics of G_U, we shall split the sum over n. In this case, the situation is slightly more complicated, because we also need to consider different cases according to whether $n-x$ is positive or negative. We therefore have three different cases:

$$\begin{aligned}
n &< x-(1+\epsilon)\tfrac{t}{\eta};\\
x-(1-\epsilon)\tfrac{t}{\eta} &< n \le x;\\
x &< n.
\end{aligned}$$

The possibility of $x-(1+\epsilon)\frac{t}{\eta} \le n \le x-(1-\epsilon)\frac{t}{\eta}$ is ruled out by Assumption 3.1.9.

We cannot consider the two sums

$$\sum_{n=0}^{\infty}\int_{-i\eta}^{0} e^{-(n+1+x)z}z^{s-1}\,\mathrm{d}z \quad , \quad \sum_{n=0}^{\infty}\int_{-i\eta}^{0} e^{-(n-x)z}z^{s-1}\,\mathrm{d}z$$

independently, since each of these sums diverges on its own. Thus, for the case $n > x$, which is the only one of the three cases to permit infinitely many values of n, we need to analyse both of these sums together. In this case, the first step involves substituting $w = (n+1+x)z$ and $w = (n-x)z$ respectively into the two integrals in the summand:

$$\begin{aligned}
&\int_{-i\eta}^{0} e^{-(n+1+x)z}z^{s-1}\,\mathrm{d}z - \int_{-i\eta}^{0} e^{-(n-x)z}z^{s-1}\,\mathrm{d}z\\
&= (n+1+x)^{-s}\int_{-i\eta(n+1+x)}^{0} e^{-w}w^{s-1}\,\mathrm{d}w - (n-x)^{-s}\int_{-i\eta(n-x)}^{0} e^{-w}w^{s-1}\,\mathrm{d}w\\
&= (n+1+x)^{-s}\int_{-i\eta(n+1+x)}^{-i\eta(n-x)} e^{-w}w^{s-1}\,\mathrm{d}w\\
&\qquad + \Big((n+1+x)^{-s} - (n-x)^{-s}\Big)\int_{-i\eta(n-x)}^{0} e^{-w}w^{s-1}\,\mathrm{d}w.
\end{aligned}$$

For the case $n < x-(1+\epsilon)\frac{t}{\eta}$, we proceed in the same way as with G_U: rewrite the integral $\int_{-i\eta}^{0}$, which contains a stationary point, as the difference of the integral $\int_{-\infty}^{-i\eta}$, which does not contain a stationary point, and the integral $\int_{-\infty}^{0}$, which can be computed explicitly.

Substituting the above into (193), we find:

$$\begin{aligned}G_B = \frac{e^{i\pi s/2}}{(2\pi)^s} \sum_{n=0}^{\left\lfloor x-\frac{t}{\eta}\right\rfloor} & \Bigg[\int_{-i\eta}^{0} e^{-(n+1+x)z} z^{s-1}\,\mathrm{d}z \\ & + \int_{-i\infty}^{-i\eta} e^{-(n-x)z} z^{s-1}\,\mathrm{d}z - \int_{-i\infty}^{0} e^{-(n-x)z} z^{s-1}\,\mathrm{d}z \Bigg] \\ + \frac{e^{i\pi s/2}}{(2\pi)^s} \sum_{n=\left\lceil x-\frac{t}{\eta}\right\rceil}^{\lfloor x\rfloor} & \Bigg[\int_{-i\eta}^{0} e^{-(n+1+x)z} z^{s-1}\,\mathrm{d}z + \int_{0}^{-i\eta} e^{-(n-x)z} z^{s-1}\,\mathrm{d}z \Bigg] \\ + \frac{e^{i\pi s/2}}{(2\pi)^s} \sum_{n=\lfloor x\rfloor+1}^{\infty} & \Bigg[(n+1+x)^{-s} \int_{-i\eta(n+1+x)}^{-i\eta(n-x)} e^{-w} w^{s-1}\,\mathrm{d}w \\ & + \Big((n+1+x)^{-s} - (n-x)^{-s} \Big) \int_{-i\eta(n-x)}^{0} e^{-w} w^{s-1}\,\mathrm{d}w \Bigg]. \end{aligned} \tag{194}$$

The explicit term in this sum is given by

$$-\int_{-i\infty}^{0} e^{(x-n)z} z^{s-1}\,\mathrm{d}z = e^{-i\pi s}(x-n)^{-s}\Gamma(s), \tag{195}$$

where we have used Jordan's lemma and the assumption that $x-n$ is positive.

For the other six integral terms in (194) – which become five because the 1st and 4th are identical – we use again repeated integration by parts:

$$\int_{-i\eta}^{0} e^{-(x+n+1)z} z^{s-1}\,\mathrm{d}z = \sum_{j=0}^{N-1} e^{-(x+n+1)z+it\log z} \left(\frac{1}{x+n+1-\frac{it}{z}} \cdot \frac{\mathrm{d}}{\mathrm{d}z}\right)^j \left(\frac{z^{\sigma-1}}{x+n+1-\frac{it}{z}}\right)\Bigg|_{z=-i\eta} + \int_{-i\eta}^{0} e^{-(x+n+1)z+it\log z} D_N(z; x+n+1; \sigma, t)\,\mathrm{d}z; \tag{196}$$

$$\int_{-i\infty}^{-i\eta} e^{(x-n)z} z^{s-1}\,\mathrm{d}z = -\sum_{j=0}^{N-1} e^{(x-n)z+it\log z} \left(\frac{1}{n-x-\frac{it}{z}} \cdot \frac{\mathrm{d}}{\mathrm{d}z}\right)^j \left(\frac{z^{\sigma-1}}{n-x-\frac{it}{z}}\right)\Bigg|_{z=-i\eta} + \int_{-i\infty}^{-i\eta} e^{(x-n)z+it\log z} D_N(z; n-x; \sigma, t)\,\mathrm{d}z; \tag{197}$$

$$\int_{0}^{-i\eta} e^{(x-n)z} z^{s-1}\,\mathrm{d}z = -\sum_{j=0}^{N-1} e^{(x-n)z+it\log z} \left(\frac{1}{n-x-\frac{it}{z}} \cdot \frac{\mathrm{d}}{\mathrm{d}z}\right)^j \left(\frac{z^{\sigma-1}}{n-x-\frac{it}{z}}\right)\Bigg|_{z=-i\eta} + \int_{0}^{-i\eta} e^{(x-n)z+it\log z} D_N(z; n-x; \sigma, t)\,\mathrm{d}z; \tag{198}$$

$$\int_{-i\eta(n+1+x)}^{-i\eta(n-x)} e^{-w} w^{s-1}\,\mathrm{d}w = \sum_{j=0}^{N-1} \left[e^{-w+it\log w} \left(\frac{1}{1-\frac{it}{w}} \cdot \frac{\mathrm{d}}{\mathrm{d}w}\right)^j \left(\frac{w^{\sigma-1}}{1-\frac{it}{w}}\right)\right]_{w=-i\eta(n-x)}^{w=-i\eta(n+1+x)} + \int_{-i\eta(n+1+x)}^{-i\eta(n-x)} e^{-w+it\log w} D_N(w; 1; \sigma, t)\,\mathrm{d}w; \tag{199}$$

$$\int_{-i\eta(n-x)}^{0} e^{-w} w^{s-1}\,\mathrm{d}w = \sum_{j=0}^{N-1} e^{-w+it\log w} \left(\frac{1}{1-\frac{it}{w}} \cdot \frac{\mathrm{d}}{\mathrm{d}w}\right)^j \left(\frac{w^{\sigma-1}}{1-\frac{it}{w}}\right)\Bigg|_{w=-i\eta(n-x)} + \int_{-i\eta(n-x)}^{0} e^{-w+it\log w} D_N(w; 1; \sigma, t)\,\mathrm{d}w. \tag{200}$$

Lemma 3.1.15. *For $n \leq x$, the remainder term in* (196) *can be uniformly approximated by*

$$O\big((2N+1)!!(N+1)^2 e^{\pi t/2} \sigma^{-1} t^{\sigma-N-1} (x+n+1)^{-\sigma}\big).$$

Proof. Let ${}_BI_1$ denote the expression we need to estimate, i.e.

$${}_BI_1 \coloneqq \int_{-i\eta}^{0} e^{-(x+n+1)z+it\log z} D_N(z; x+n+1; \sigma, t)\,\mathrm{d}z.$$

By Lemma 3.1.10, we have

$$D_N(z; x+n+1; \sigma, t) = O\Big((2N-1)!!(N+1)^2 |z|^{\sigma-1} \max\big((x+n+1)|z|, t\big)^{-N}\Big),$$

and therefore

$$
\begin{aligned}
{}_BI_1 &= O\bigg(\int_{-i\eta}^{0} e^{-(x+n+1)z+it\log z}(2N-1)!!(N+1)^2|z|^{\sigma-1}\max\big((x+n+1)|z|,t\big)^{-N}\,\mathrm{d}z\bigg)\\
&= O\bigg((2N-1)!!(N+1)^2e^{\pi t/2}\int_{-i\eta}^{0}|z|^{\sigma-1}\max((x+n+1)|z|,t)^{-N}\,\mathrm{d}z\bigg).
\end{aligned}
$$

If $(x+n+1)\eta \le t$, the above yields

$$
\begin{aligned}
{}_BI_1 &= O\bigg((2N-1)!!(N+1)^2e^{\pi t/2}\int_{-i\eta}^{0}|z|^{\sigma-1}t^{-N}\,\mathrm{d}z\bigg)\\
&= O\Big((2N-1)!!(N+1)^2e^{\pi t/2}\big(\tfrac{\eta^\sigma}{\sigma}\big)t^{-N}\Big)\\
&= O\Big((2N-1)!!(N+1)^2e^{\pi t/2}\sigma^{-1}t^{\sigma-N}(x+n+1)^{-\sigma}\Big).
\end{aligned}
$$

If $(x+n+1)\eta > t$, we find

$$
\begin{aligned}
{}_BI_1 &= O\Bigg((2N-1)!!(N+1)^2e^{\pi t/2}\bigg(\int_{\eta}^{t/(x+n+1)}(x+n+1)^{-N}z^{\sigma-N-1}\,\mathrm{d}z\\
&\qquad\qquad + \int_{t/(x+n+1)}^{0} z^{\sigma-1}t^{-N}\,\mathrm{d}z\bigg)\Bigg)\\
&= O\Bigg((2N-1)!!(N+1)^2e^{\pi t/2}\bigg[\tfrac{(x+n+1)^{-N}}{\sigma-N}\Big(\eta^{\sigma-N}-\big(\tfrac{t}{x+n+1}\big)^{\sigma-N}\Big)+\tfrac{t^{-N}}{\sigma}\big(\tfrac{t}{x+n+1}\big)^{\sigma}\bigg]\Bigg)\\
&= O\Bigg((2N-1)!!(N+1)^2e^{\pi t/2}\bigg[\frac{(x+n+1)^{\sigma-N}\eta^{\sigma-N}}{(N-\sigma)(x+n+1)^{\sigma}}+\frac{N}{N-\sigma}\cdot\frac{t^{\sigma-N}}{\sigma(x+n+1)^{\sigma}}\bigg]\Bigg)\\
&= O\Big((2N-1)!!(N+1)^2e^{\pi t/2}\sigma^{-1}t^{\sigma-N}(x+n+1)^{-\sigma}\Big),
\end{aligned}
$$

where the O-bound is uniform provided $N \ge 2$. Thus, in both cases,

$$
{}_BI_1 = O\Big((2N-1)!!(N+1)^2e^{\pi t/2}\sigma^{-1}t^{\sigma-N}(x+n+1)^{-\sigma}\Big). \tag{201}
$$

But here the exponent of η is just one too big. So we apply integration by parts once more to the original expression for ${}_BI_1$:

$$
\begin{aligned}
{}_BI_1 = -\bigg[e^{-(x+n+1)z+it\log z}\Big(\tfrac{1}{x+n+1-\frac{it}{z}}\Big)D_N(z;x+n+1;\sigma,t)\bigg]_{-i\eta}^{0}&\\
+\int_{-i\eta}^{0}e^{-(x+n+1)z+it\log z}D_{N+1}(z;x+n+1;\sigma,t)\,\mathrm{d}z.&
\end{aligned}
$$

Using the results of Lemma 3.1.10 for the first half of this expression, and equation (201) (with N replaced by $N+1$, so that our $N \ge 2$ assumption becomes only $N \ge 1$) for the second half, we find:

$$
\begin{aligned}
{}_BI_1 &= O\Big(\tfrac{e^{\pi t/2}z}{(x+n+1)z-it}(2N-1)!!(N+1)^2|z|^{\sigma-1}\max\big((x+n+1)|z|,t\big)^{-N}\Big|_{z=-i\eta}\Big)\\
&\qquad\qquad\qquad\qquad + O\Big((2N+1)!!(N+2)^2e^{\pi t/2}\sigma^{-1}t^{\sigma-N-1}(x+n+1)^{-\sigma}\Big)\\
&= O\Big((2N+1)!!(N+1)e^{\pi t/2}\eta^{\sigma}\Big(\tfrac{\max((x+n+1)\eta,t)^{-N}}{(x+n+1)\eta+t}\Big)\Big)\\
&\qquad\qquad\qquad\qquad + O\Big((2N+1)!!(N+1)^2e^{\pi t/2}\sigma^{-1}t^{\sigma-N-1}(x+n+1)^{-\sigma}\Big).
\end{aligned}
$$

If $(x+n+1)\eta \le t$, the above yields

$$
\begin{aligned}
{}_BI_1 &= O\Big((2N+1)!!(N+1)e^{\pi t/2}\Big[\eta^{\sigma}\cdot\tfrac{t^{-N}}{t} + (N+1)\sigma^{-1}t^{\sigma-N-1}(x+n+1)^{-\sigma}\Big]\Big)\\
&= O\Big((2N+1)!!(N+1)e^{\pi t/2}\Big[\big(\tfrac{t}{x+n+1}\big)^{\sigma}t^{-N-1} + (N+1)\sigma^{-1}t^{\sigma-N-1}(x+n+1)^{-\sigma}\Big]\Big)\\
&= O\big((2N+1)!!(N+1)^2e^{\pi t/2}\sigma^{-1}t^{\sigma-N-1}(x+n+1)^{-\sigma}\big).
\end{aligned}
$$

If $(x+n+1)\eta > t$, we find

$$
\begin{aligned}
{}_BI_1 &= O\Big((2N+1)!!(N+1)e^{\pi t/2}\Big[\eta^{\sigma}\cdot\tfrac{(x+n+1)^{-N}\eta^{-N}}{(x+n+1)\eta} + (N+1)\sigma^{-1}t^{\sigma-N-1}(x+n+1)^{-\sigma}\Big]\Big)\\
&= O\Big((2N+1)!!(N+1)e^{\pi t/2}\Big[\big(\tfrac{t}{x+n+1}\big)^{\sigma-N-1}(x+n+1)^{-N-1}\\
&\qquad\qquad\qquad\qquad\qquad + (N+1)\sigma^{-1}t^{\sigma-N-1}(x+n+1)^{-\sigma}\Big]\Big)\\
&= O\big((2N+1)!!(N+1)^2e^{\pi t/2}\sigma^{-1}t^{\sigma-N-1}(x+n+1)^{-\sigma}\big).
\end{aligned}
$$

Thus, in both cases, we have the required estimate for ${}_BI_1$, and the O-bound is uniform in all parameters. □

Lemma 3.1.16. *For $n < x-(1+\epsilon)\frac{t}{\eta}$, the remainder term in* (197) *can be uniformly approximated by*

$$O\big((2N+1)!!(N+1)e^{\pi t/2}\eta^{\sigma-N-1}(x-n)^{-N-1}\epsilon^{-2N-2}(1+\epsilon)^{2N+2}\big).$$

For $x-(1-\epsilon)\frac{t}{\eta} < n \le x$, the remainder term in (198) *can be uniformly approximated by*

$$O\big((2N+1)!!(N+1)^2e^{\pi t/2}\sigma^{-1}\eta^{\sigma}t^{-N-1}\epsilon^{-2N-2}\big).$$

Proof. Let ${}_BI_2$ and ${}_BI_3$ denote the two expressions we need to estimate, i.e.

$${}_BI_2 \coloneqq \int_{-i\infty}^{-i\eta} e^{(x-n)z+it\log z}D_N(z;n-x;\sigma,t)\,\mathrm{d}z$$

and

$$
{}_BI_3 \coloneqq \int_0^{-i\eta} e^{(x-n)z+it\log z} D_N(z;n-x;\sigma,t)\,\mathrm{d}z.
$$

For ${}_BI_2$, we can use the same argument as in Case 2 of Lemma 3.1.12 to estimate D_N. Starting from the expression (176) and using the fact that $|(n-x)z| \geq (x-n)\eta > (1+\epsilon)t$, we find

$$
\begin{aligned}
D_N(z;n-x;\sigma,t) &= O\bigg((2N-1)!!|z|^{\sigma-1}\sum_{b=0}^{N}\sum_{c=0}^{N}\Big|\tfrac{t}{(n-x)z}\Big|^b|\sigma|^c\Big|\tfrac{(n-x)z}{((n-x)z-it)^2}\Big|^N\bigg)\\
&= O\bigg((2N-1)!!(N+1)^2|z|^{\sigma-1}\Big|\tfrac{(n-x)z}{((n-x)z-it)^2}\Big|^N\bigg).
\end{aligned}
$$

Since $(n-x)z$ is positive imaginary with modulus at least $(1+\epsilon)t$, it follows that

$$
\big|(n-x)z-it\big| > \tfrac{\epsilon}{1+\epsilon}\big|(n-x)z\big|,
$$

and thus

$$
\begin{aligned}
D_N &= O\Big((2N-1)!!(N+1)^2|z|^{\sigma-1}\big|(n-x)z\big|^{-N}\big(\tfrac{\epsilon}{1+\epsilon}\big)^{-2N}\Big)\\
&= O\big((2N-1)!!(N+1)^2|z|^{\sigma-N-1}(x-n)^{-N}\epsilon^{-2N}(1+\epsilon)^{2N}\big).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
{}_BI_2 &= O\bigg(\int_{-i\infty}^{-i\eta} e^{(x-n)z+it\log z}(2N-1)!!(N+1)^2|z|^{\sigma-N-1}(x-n)^{-N}\epsilon^{-2N}(1+\epsilon)^{2N}\,\mathrm{d}z\bigg)\\
&= O\big((2N-1)!!(N+1)e^{\pi t/2}\eta^{\sigma-N}(x-n)^{-N}\epsilon^{-2N}(1+\epsilon)^{2N}\big),
\end{aligned}
$$

where the O-bound is uniform provided $N \geq 2$.

But here the exponent of η is just one too big. So we apply integration by parts once more to the original expression for ${}_BI_2$:

$$
\begin{aligned}
{}_BI_2 = -\Big[e^{(x-n)z+it\log z}\Big(\tfrac{1}{n-x-\frac{it}{z}}\Big)D_N(z;n-x;\sigma,t)\Big]_{-i\infty}^{-i\eta}&\\
+\int_{-i\infty}^{-i\eta} e^{(x-n)z+it\log z} D_{N+1}(z;n-x;\sigma,t)\,\mathrm{d}z.&
\end{aligned}
$$

Using the estimates we just derived for D_N and (with N replaced by $N+1$, so that our

$N \geq 2$ assumption becomes only $N \geq 1$) for ${}_BI_2$, this becomes

$$\begin{aligned}
{}_BI_2 &= O\bigg(\tfrac{e^{\pi t/2} z}{(n-x)z - it} (2N-1)!!(N+1)^2 |z|^{\sigma - N - 1} (x-n)^{-N} \epsilon^{-2N} (1+\epsilon)^{2N} \bigg|_{z=-i\eta} \bigg) \\
&\qquad\qquad + O\big((2N+1)!!(N+2) e^{\pi t/2} \eta^{\sigma - N - 1} (x-n)^{-N-1} \epsilon^{-2N-2} (1+\epsilon)^{2N+2} \big) \\
&= O\bigg(\tfrac{e^{\pi t/2}\eta}{\frac{\epsilon}{1+\epsilon}(x-n)\eta} (2N+1)!!(N+1) \eta^{\sigma - N - 1} (x-n)^{-N} \epsilon^{-2N} (1+\epsilon)^{2N} \bigg) \\
&\qquad\qquad + O\big((2N+1)!!(N+1) e^{\pi t/2} \eta^{\sigma - N - 1} (x-n)^{-N-1} \epsilon^{-2N-2} (1+\epsilon)^{2N+2} \big) \\
&= O\Big((2N+1)!!(N+1) e^{\pi t/2} \eta^{\sigma - N - 1} (x-n)^{-N-1} \epsilon^{-2N-1} (1+\epsilon)^{2N+1} \big[1 + \tfrac{1+\epsilon}{\epsilon} \big] \Big),
\end{aligned}$$

where we have again used the inequality $(x-n)\eta - t > \frac{\epsilon}{1+\epsilon}(x-n)\eta$. This gives us the required expression for ${}_BI_2$.

For ${}_BI_3$, we can use the same argument as in Case 1 of Lemma 3.1.12 to estimate D_N. Starting from the expression (176) and using the fact that $|(n-x)z| \leq (x-n)\eta < (1-\epsilon)t$, we find

$$\begin{aligned}
D_N(z; n-x; \sigma, t) &= O\Bigg((2N-1)!! |z|^{\sigma-1} \sum_{b=0}^{N} \sum_{c=0}^{N} \left| \tfrac{(n-x)z}{t} \right|^{N-b} |\sigma|^c \left| \tfrac{t}{((n-x)z - it)^2} \right|^N \Bigg) \\
&= O\bigg((2N-1)!!(N+1)^2 |z|^{\sigma - 1} \left| \tfrac{t}{((n-x)z-it)^2} \right|^N \bigg).
\end{aligned}$$

Since $(n-x)z$ is non-negative imaginary with modulus at most $(1-\epsilon)t$, it follows that $\big|(n-x)z - it\big| > \epsilon t$, and thus

$$\begin{aligned}
D_N &= O\Big((2N-1)!!(N+1)^2 |z|^{\sigma-1} t^N (\epsilon t)^{-2N} \Big) \\
&= O\big((2N-1)!!(N+1)^2 |z|^{\sigma-1} t^{-N} \epsilon^{-2N} \big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
{}_BI_3 &= O\bigg(\int_0^{-i\eta} e^{(x-n)z + it\log z} (2N-1)!!(N+1)^2 |z|^{\sigma-1} t^{-N} \epsilon^{-2N} \, \mathrm{d}z \bigg) \\
&= O\Big((2N-1)!!(N+1)^2 e^{\pi t/2} \big(\tfrac{\eta^\sigma}{\sigma} \big) t^{-N} \epsilon^{-2N} \Big).
\end{aligned}$$

But here the exponent of η is just one too big. So we apply integration by parts once more to the original expression for ${}_BI_3$:

$$\begin{aligned}
{}_BI_3 = -\bigg[e^{(x-n)z + it\log z} \Big(\tfrac{1}{n-x-\frac{it}{z}} \Big) D_N(z; n-x; \sigma, t) \bigg]_0^{-i\eta} \\
+ \int_0^{-i\eta} e^{(x-n)z + it\log z} D_{N+1}(z; n-x; \sigma, t) \, \mathrm{d}z.
\end{aligned}$$

Using the estimates we just derived for D_N and (with N replaced by $N+1$) for ${}_BI_3$, this becomes

$$\begin{aligned} {}_BI_3 &= O\left(\tfrac{e^{\pi t/2} z}{(n-x)z - it} (2N-1)!!(N+1)^2 |z|^{\sigma-1} t^{-N} \epsilon^{-2N} \Big|_{z=-i\eta} \right) \\ &\qquad\qquad + O\big((2N+1)!!(N+2)^2 e^{\pi t/2} \sigma^{-1} \eta^{\sigma} t^{-N-1} \epsilon^{-2N-2}\big) \\ &= O\left(\tfrac{e^{\pi t/2}\eta}{\epsilon t} (2N+1)!!(N+1) \eta^{\sigma-1} t^{-N} \epsilon^{-2N} \right) \\ &\qquad\qquad + O\big((2N+1)!!(N+1)^2 e^{\pi t/2} \sigma^{-1} \eta^{\sigma} t^{-N-1} \epsilon^{-2N-2}\big) \\ &= O\big((2N+1)!!(N+1)^2 e^{\pi t/2} \sigma^{-1} \eta^{\sigma} t^{-N-1} \epsilon^{-2N-2}\big), \end{aligned}$$

where we have used again the inequality $\big|(n-x)z - it\big| > \epsilon t$. This gives us the required expression for ${}_BI_3$. $\square$

Lemma 3.1.17. *For $n > x$, the remainder term in* (199) *can be uniformly approximated by*

$$O\big((2N+1)!!(N+1) e^{\pi t/2} \eta^{\sigma-N-1} (n-x)^{\sigma-N-1}\big),$$

and the remainder term in (200) *can be uniformly approximated by*

$$O\big((2N+1)!!(N+1)^2 e^{\pi t/2} \sigma^{-1} t^{\sigma-N-1}\big).$$

Proof. Let ${}_BI_4$ and ${}_BI_5$ denote the two expressions we need to estimate, i.e.

$${}_BI_4 \coloneqq \int_{-i\eta(n+1+x)}^{-i\eta(n-x)} e^{-w+it\log w} D_N(w;1;\sigma,t)\,\mathrm{d}w$$

and

$${}_BI_5 \coloneqq \int_{-i\eta(n-x)}^{0} e^{-w+it\log w} D_N(w;1;\sigma,t)\,\mathrm{d}w.$$

By the first half (180) of Lemma 3.1.10, we have

$$D_N(w;1;\sigma,t) = O\big((2N-1)!!(N+1)^2 |w|^{\sigma-N-1}\big) \tag{202}$$

for $\mathrm{Im}(w) < 0$, which holds here since we have assumed $n - x > 0$. Thus,

$$\begin{aligned} {}_BI_4 &= O\left(\int_{-i\eta(n+1+x)}^{-i\eta(n-x)} e^{-w+it\log w} (2N-1)!!(N+1)^2 |w|^{\sigma-N-1}\,\mathrm{d}w \right) \\ &= O\Big((2N-1)!!(N+1) e^{\pi t/2} \eta^{\sigma-N} \big[(n-x)^{\sigma-N} - (n+1+x)^{\sigma-N}\big] \Big), \end{aligned} \tag{203}$$

where the O-bound is uniform provided $N \geq 2$.

But here the exponent of η is just one too big. So we apply integration by parts once more to the original expression for ${}_BI_4$:

$$
{}_BI_4 = -\Big[e^{-w+it\log w}\Big(\tfrac{1}{1-\frac{it}{w}}\Big)D_N(w;1;\sigma,t)\Big]_{-i\eta(n+1+x)}^{-i\eta(n-x)} \\ + \int_{-i\eta(n+1+x)}^{-i\eta(n-x)} e^{-w+it\log w}D_{N+1}(w;1;\sigma,t)\,\mathrm{d}w.
$$

Using equation (202) for the first half of this expression, and equation (203) (with N replaced by $N+1$, so that our $N \geq 2$ assumption becomes only $N \geq 1$) for the second half, we find:

$$
\begin{aligned}
{}_BI_4 &= O\Big(e^{\pi t/2}\big[(2N-1)!!(N+1)^2|w|^{\sigma-N-1}\big]_{-i\eta(n+1+x)}^{-i\eta(n-x)}\Big) \\
&\qquad + O\Big((2N+1)!!(N+2)e^{\pi t/2}\eta^{\sigma-N-1}\big[(n-x)^{\sigma-N-1}-(n+1+x)^{\sigma-N-1}\big]\Big) \\
&= O\Big((2N+1)!!(N+1)e^{\pi t/2}\eta^{\sigma-N-1}\big[(n-x)^{\sigma-N-1}-(n+1+x)^{\sigma-N-1}\big]\Big),
\end{aligned}
$$

which gives the required expression for ${}_BI_4$.

${}_BI_5$ is slightly harder to estimate, since we need to split into two separate cases. By Lemma 3.1.10, we have

$$
D_N(w;1;\sigma,t) = O\Big((2N-1)!!(N+1)^2|w|^{\sigma-1}\max\big(|w|,t\big)^{-N}\Big)
$$

for $\mathrm{Im}(w) < 0$, which again holds here by assumption. Thus,

$$
\begin{aligned}
{}_BI_5 &= O\bigg(\int_{-i\eta(n-x)}^{0} e^{-w+it\log w}(2N-1)!!(N+1)^2|w|^{\sigma-1}\max\big(|w|,t\big)^{-N}\,\mathrm{d}w\bigg) \\
&= O\bigg((2N-1)!!(N+1)^2e^{\pi t/2}\int_{\eta(n-x)}^{0} w^{\sigma-1}\max(w,t)^{-N}\,\mathrm{d}w\bigg).
\end{aligned}
$$

Case 1: $\boldsymbol{\eta(n-x) \leq t}$.

In this case,

$$
\begin{aligned}
{}_BI_5 &= O\bigg((2N-1)!!(N+1)^2e^{\pi t/2}\int_{\eta(n-x)}^{0} w^{\sigma-1}t^{-N}\,\mathrm{d}w\bigg) \\
&= O\Big((2N-1)!!(N+1)^2e^{\pi t/2}\sigma^{-1}\eta^{\sigma}t^{-N}(n-x)^{\sigma}\Big). \qquad (204)
\end{aligned}
$$

Again the exponent of η is just one too big, so we apply integration by parts once more to the original expression for ${}_BI_5$:

$$
{}_BI_5 = \Big[e^{-w+it\log w}\Big(\tfrac{1}{1-\frac{it}{w}}\Big)D_N(w;1;\sigma,t)\Big]_{w=-i\eta(n-x)} + \int_{\eta(n-x)}^{0} e^{-w+it\log w}D_{N+1}(w;1;\sigma,t)\,\mathrm{d}w.
$$

Using (181) from Lemma 3.1.10 for the first half of this expression, and (204) (with N

replaced by $N+1$) for the second half, we find:

$$\begin{aligned}
{}_BI_5 &= O\Big(e^{\pi t/2}\Big(\tfrac{\eta(n-x)}{t}\Big)(2N-1)!!(N+1)^2[\eta(n-x)]^{\sigma-1}t^{-N}\Big) \\
&\qquad\qquad\qquad + O\Big((2N+1)!!(N+2)^2e^{\pi t/2}\sigma^{-1}\eta^{\sigma}t^{-N-1}(n-x)^{\sigma}\Big) \\
&= O\Big((2N+1)!!(N+1)^2e^{\pi t/2}\sigma^{-1}[\eta(n-x)]^{\sigma}t^{-N-1}\Big) \\
&= O\big((2N+1)!!(N+1)^2e^{\pi t/2}\sigma^{-1}t^{\sigma-N-1}\big).
\end{aligned}$$

Case 2: $\boldsymbol{\eta(n-x)>t}$.

In this case,

$$\begin{aligned}
{}_BI_5 &= O\Bigg((2N-1)!!(N+1)^2e^{\pi t/2}\Bigg[\int_{\eta(n-x)}^{t} w^{\sigma-N-1}\,\mathrm{d}w + \int_t^0 w^{\sigma-1}t^{-N}\,\mathrm{d}w\Bigg]\Bigg) \\
&= O\Big((2N-1)!!(N+1)e^{\pi t/2}\Big[t^{\sigma-N} - \big(\eta(x-n)\big)^{\sigma-N} + (N+1)\sigma^{-1}t^{\sigma-N}\Big]\Big) \\
&= O\Big((2N-1)!!(N+1)^2e^{\pi t/2}\sigma^{-1}t^{\sigma-N}\Big), \qquad (205)
\end{aligned}$$

where the O-bound is uniform provided $N\geq 2$. Again the exponent of η is just one too big, so we apply integration by parts as before. Using (180) from Lemma 3.1.10 for the first half of the resulting expression and (205) (with N replaced by $N+1$, so that our $N\geq 2$ assumption becomes only $N\geq 1$) for the second half, we find:

$$\begin{aligned}
{}_BI_5 &= O\Big(e^{\pi t/2}(2N-1)!!(N+1)^2[\eta(n-x)]^{\sigma-N-1}\Big) \\
&\qquad\qquad\qquad + O\Big((2N+1)!!(N+2)^2e^{\pi t/2}\sigma^{-1}t^{\sigma-N-1}\Big) \\
&= O\big((2N+1)!!(N+1)^2e^{\pi t/2}\sigma^{-1}t^{\sigma-N-1}\big).
\end{aligned}$$

In both cases, we have the required estimate for ${}_BI_5$. □

Lemma 3.1.18. *We have the following uniform estimate for G_B:*

$$\begin{aligned}
G_B(\sigma,t;\eta;x) &= \frac{e^{-i\pi s/2}}{(2\pi)^s}\sum_{n=0}^{\lfloor x-\frac{t}{\eta}\rfloor}\frac{\Gamma(s)}{(x-n)^s} \\
&\quad - \frac{e^{i\pi\sigma/2}e^{it\log\eta}}{(2\pi)^s}\sum_{n=0}^{M}\sum_{j=0}^{N-1}e^{i(n-x)\eta}\Bigg[\Big(\tfrac{1}{n-x-\frac{it}{z}}\cdot\tfrac{\mathrm{d}}{\mathrm{d}z}\Big)^j\Big(\tfrac{z^{\sigma-1}}{n-x-\frac{it}{z}}\Big)\Bigg]_{z=-i\eta} \\
&\quad + \frac{e^{i\pi\sigma/2}e^{it\log\eta}}{(2\pi)^s}\sum_{n=0}^{M}\sum_{j=0}^{N-1}e^{i(x+n+1)\eta}\Bigg[\Big(\tfrac{1}{x+n+1-\frac{it}{z}}\cdot\tfrac{\mathrm{d}}{\mathrm{d}z}\Big)^j\Big(\tfrac{z^{\sigma-1}}{x+n+1-\frac{it}{z}}\Big)\Bigg]_{z=-i\eta} \\
&\qquad + O\Big((2N+1)!!(N+1)^2\sigma^{-1}\min(t,\eta)^{\sigma-N-1}\big[x^{1-\sigma} \\
&\qquad\qquad + (x-\lfloor x\rfloor)^{-N-1}\big(\tfrac{1+\epsilon}{\epsilon}\big)^{2N+2} + x^{-\sigma}\big(\lfloor x\rfloor - x+1\big)^{-N-1}\big]\Big),
\end{aligned}$$

where M is a finite number depending only on N, x, and η, or $M = \infty$ if $\eta \in 2\pi\mathbb{Z}$.

Proof. We use Lemma 3.1.15 to establish that for $n \le x$, equation (196) becomes:

$$\int_{-i\eta}^{0} e^{-(x+n+1)z} z^{s-1}\,\mathrm{d}z = e^{\pi t/2} e^{it\log\eta} \sum_{j=0}^{N-1} e^{i(x+n+1)\eta} \left[\left(\frac{1}{x+n+1-\frac{it}{z}} \cdot \frac{\mathrm{d}}{\mathrm{d}z} \right)^j \left(\frac{z^{\sigma-1}}{x+n+1-\frac{it}{z}} \right) \right]_{z=-i\eta}$$
$$+ O\big((2N+1)!!(N+1)^2 e^{\pi t/2} \sigma^{-1} t^{\sigma-N-1} (x+n+1)^{-\sigma}\big). \quad (206)$$

We use Lemma 3.1.16 to establish that for $n < x - (1+\epsilon)\frac{t}{\eta}$ and $x - (1-\epsilon)\frac{t}{\eta} < n \le x$ respectively, equations (197) and (198) become:

$$\int_{-i\infty}^{-i\eta} e^{(x-n)z} z^{s-1}\,\mathrm{d}z = -e^{\pi t/2} e^{it\log\eta} \sum_{j=0}^{N-1} e^{-i(x-n)\eta} \left[\left(\frac{1}{n-x-\frac{it}{z}} \cdot \frac{\mathrm{d}}{\mathrm{d}z} \right)^j \left(\frac{z^{\sigma-1}}{n-x-\frac{it}{z}} \right) \right]_{z=-i\eta}$$
$$+ O\big((2N+1)!!(N+1) e^{\pi t/2} \eta^{\sigma-N-1} (x-n)^{-N-1} \epsilon^{-2N-2} (1+\epsilon)^{2N+2}\big) \quad (207)$$

and

$$\int_{0}^{-i\eta} e^{(x-n)z} z^{s-1}\,\mathrm{d}z = -e^{\pi t/2} e^{it\log\eta} \sum_{j=0}^{N-1} e^{-i(x-n)\eta} \left(\frac{1}{n-x-\frac{it}{z}} \cdot \frac{\mathrm{d}}{\mathrm{d}z} \right)^j \left(\frac{z^{\sigma-1}}{n-x-\frac{it}{z}} \right) \Bigg|_{z=-i\eta}$$
$$+ O\big((2N+1)!!(N+1)^2 e^{\pi t/2} \sigma^{-1} \eta^{\sigma} t^{-N-1} \epsilon^{-2N-2}\big). \quad (208)$$

We use Lemma 3.1.17 to establish that for $n > x$, equations (199) and (200) become:

$$\int_{-i\eta(n+1+x)}^{-i\eta(n-x)} e^{-w} w^{s-1}\,\mathrm{d}w$$
$$= e^{\pi t/2} e^{it\log\eta} \sum_{j=0}^{N-1} e^{i(n+1+x)\eta} (n+1+x)^{it} \left[\left(\frac{1}{1-\frac{it}{w}} \cdot \frac{\mathrm{d}}{\mathrm{d}w} \right)^j \left(\frac{w^{\sigma-1}}{1-\frac{it}{w}} \right) \right]_{w=-i\eta(n+1+x)}$$
$$- e^{\pi t/2} e^{it\log\eta} \sum_{j=0}^{N-1} e^{i(n-x)\eta} (n-x)^{it} \left[\left(\frac{1}{1-\frac{it}{w}} \cdot \frac{\mathrm{d}}{\mathrm{d}w} \right)^j \left(\frac{w^{\sigma-1}}{1-\frac{it}{w}} \right) \right]_{w=-i\eta(n-x)}$$
$$+ O\big((2N+1)!!(N+1) e^{\pi t/2} \eta^{\sigma-N-1} (n-x)^{\sigma-N-1}\big)$$

and

$$\int_{-i\eta(n-x)}^{0} e^{-w} w^{s-1}\,\mathrm{d}w = e^{\pi t/2} e^{it\log\eta} \sum_{j=0}^{N-1} e^{i(n-x)\eta} (n-x)^{it} \left[\left(\frac{1}{1-\frac{it}{w}} \cdot \frac{\mathrm{d}}{\mathrm{d}w} \right)^j \left(\frac{w^{\sigma-1}}{1-\frac{it}{w}} \right) \right]_{w=-i\eta(n-x)}$$
$$+ O\big((2N+1)!!(N+1)^2 e^{\pi t/2} \sigma^{-1} t^{\sigma-N-1}\big).$$

Now, re-substituting $w = (n+1+x)z$ or $w = (n-x)z$ as appropriate, the expression for equation (199) becomes:

$$\begin{aligned}
&e^{\pi t/2}e^{it\log\eta}\sum_{j=0}^{N-1}e^{i(n+1+x)\eta}(n+1+x)^{it}\left[\left(\frac{1/(n+1+x)}{1-\frac{it}{(n+1+x)z}}\cdot\frac{\mathrm{d}}{\mathrm{d}z}\right)^{j}\left(\frac{((n+1+x)z)^{\sigma-1}}{1-\frac{it}{(n+1+x)z}}\right)\right]_{z=-i\eta}\\
&\qquad-e^{\pi t/2}e^{it\log\eta}\sum_{j=0}^{N-1}e^{i(n-x)\eta}(n-x)^{it}\left[\left(\frac{1/(n-x)}{1-\frac{it}{(n-x)z}}\cdot\frac{\mathrm{d}}{\mathrm{d}z}\right)^{j}\left(\frac{((n-x)z)^{\sigma-1}}{1-\frac{it}{(n-x)z}}\right)\right]_{z=-i\eta}\\
&\qquad\qquad+O\big((2N+1)!!(N+1)e^{\pi t/2}\eta^{\sigma-N-1}(n-x)^{\sigma-N-1}\big)\\
&=e^{\pi t/2}e^{it\log\eta}(n+1+x)^{s}\sum_{j=0}^{N-1}e^{i(n+1+x)\eta}\left[\left(\frac{1}{n+1+x-\frac{it}{z}}\cdot\frac{\mathrm{d}}{\mathrm{d}z}\right)^{j}\left(\frac{(z)^{\sigma-1}}{n+1+x-\frac{it}{z}}\right)\right]_{z=-i\eta}\\
&\qquad-e^{\pi t/2}e^{it\log\eta}(n-x)^{s}\sum_{j=0}^{N-1}e^{i(n-x)\eta}\left[\left(\frac{1}{n-x-\frac{it}{z}}\cdot\frac{\mathrm{d}}{\mathrm{d}z}\right)^{j}\left(\frac{z^{\sigma-1}}{n-x-\frac{it}{z}}\right)\right]_{z=-i\eta}\\
&\qquad\qquad+O\big((2N+1)!!(N+1)e^{\pi t/2}\eta^{\sigma-N-1}(n-x)^{\sigma-N-1}\big).
\end{aligned}\tag{209}$$

Similarly, after re-substituting $w=(n-x)z$, the expression for equation (200) becomes

$$\begin{aligned}
&e^{\pi t/2}e^{it\log\eta}(n-x)^{s}\sum_{j=0}^{N-1}e^{i(n-x)\eta}\left[\left(\frac{1}{n-x-\frac{it}{z}}\cdot\frac{\mathrm{d}}{\mathrm{d}z}\right)^{j}\left(\frac{z^{\sigma-1}}{n-x-\frac{it}{z}}\right)\right]_{z=-i\eta}\\
&\qquad\qquad+O\big((2N+1)!!(N+1)^2e^{\pi t/2}\sigma^{-1}t^{\sigma-N-1}\big).
\end{aligned}\tag{210}$$

Substituting (195), (206), (207), (208), (209), (210) into the expression (194) for G_B, and noting the cancellation of certain terms originating from (209) and (210), we find:

$$
\begin{aligned}
G_B = \frac{e^{-i\pi s/2}}{(2\pi)^s} &\sum_{n=0}^{\left\lfloor x-\frac{t}{\eta}\right\rfloor} \frac{\Gamma(s)}{(x-n)^s} \\
&+ \frac{e^{i\pi\sigma/2}}{(2\pi)^s} \sum_{n=0}^{\lfloor x\rfloor} \left(e^{it\log\eta} \sum_{j=0}^{N-1} e^{i(x+n+1)\eta} \left[\left(\frac{1}{x+n+1-\frac{it}{z}} \cdot \frac{\mathrm{d}}{\mathrm{d}z} \right)^j \left(\frac{z^{\sigma-1}}{x+n+1-\frac{it}{z}} \right) \right]_{z=-i\eta} \right) \\
&- \frac{e^{i\pi\sigma/2}}{(2\pi)^s} \sum_{n=0}^{\lfloor x\rfloor} \left(e^{it\log\eta} \sum_{j=0}^{N-1} e^{-i(x-n)\eta} \left[\left(\frac{1}{n-x-\frac{it}{z}} \cdot \frac{\mathrm{d}}{\mathrm{d}z} \right)^j \left(\frac{z^{\sigma-1}}{n-x-\frac{it}{z}} \right) \right]_{z=-i\eta} \right) \\
&+ \frac{e^{i\pi\sigma/2}}{(2\pi)^s} \sum_{n=\lfloor x\rfloor+1}^{\infty} \left(e^{it\log\eta} \sum_{j=0}^{N-1} e^{i(n+1+x)\eta} \left[\left(\frac{1}{n+1+x-\frac{it}{z}} \cdot \frac{\mathrm{d}}{\mathrm{d}z} \right)^j \left(\frac{(z)^{\sigma-1}}{n+1+x-\frac{it}{z}} \right) \right]_{z=-i\eta} \right) \\
&- \frac{e^{i\pi\sigma/2}}{(2\pi)^s} \sum_{n=\lfloor x\rfloor+1}^{\infty} \left(e^{it\log\eta} \sum_{j=0}^{N-1} e^{i(n-x)\eta} \left[\left(\frac{1}{n-x-\frac{it}{z}} \cdot \frac{\mathrm{d}}{\mathrm{d}z} \right)^j \left(\frac{z^{\sigma-1}}{n-x-\frac{it}{z}} \right) \right]_{z=-i\eta} \right) \\
&+ O\Bigg(\sum_{n=0}^{\lfloor x\rfloor} (2N+1)!!(N+1)^2\sigma^{-1}t^{\sigma-N-1}(x+n+1)^{-\sigma} \Bigg) \\
&+ O\Bigg(\sum_{n=0}^{\left\lfloor x-\frac{t}{\eta}\right\rfloor} (2N+1)!!(N+1)\eta^{\sigma-N-1}(x-n)^{-N-1}\epsilon^{-2N-2}(1+\epsilon)^{2N+2} \Bigg) \\
&+ O\Bigg(\sum_{n=\left\lceil x-\frac{t}{\eta}\right\rceil}^{\lfloor x\rfloor} (2N+1)!!(N+1)^2\sigma^{-1}\eta^{\sigma}t^{-N-1}\epsilon^{-2N-2} \Bigg) \\
&+ O\Bigg(\sum_{n=\lfloor x\rfloor+1}^{\infty} (n+1+x)^{-s}(2N+1)!!(N+1)\eta^{\sigma-N-1}(n-x)^{\sigma-N-1} \Bigg) \\
&+ O\Bigg(\sum_{n=\lfloor x\rfloor+1}^{\infty} \Big((n+1+x)^{-s} - (n-x)^{-s} \Big) (2N+1)!!(N+1)^2\sigma^{-1}t^{\sigma-N-1} \Bigg).
\end{aligned}
$$

Let us consider each of the remainder terms in turn, with the O-bound in each case being uniform. First,

$$\sum_{n=0}^{\lfloor x\rfloor}(x+n+1)^{-\sigma} = O\big(x^{1-\sigma}\big);$$

this follows by splitting the series into sums from $\frac{x}{2^k}$ to $\frac{x}{2^{k-1}}$ for $1 \le k \le \log_2 x$. Second,

$$\sum_{n=0}^{\left\lfloor x-\frac{t}{\eta}\right\rfloor} (x-n)^{-N-1} = O\big((x-\lfloor x\rfloor)^{-N-1}\big) + O(1).$$

Third, if $\frac{t}{\eta} < 1$, the sum

$$\sum_{n=\left\lceil x-\frac{t}{\eta}\right\rceil}^{\lfloor x\rfloor} 1$$

is non-existent, while if $\frac{t}{\eta} \geq 1$ it is $O\big(\frac{t}{\eta}\big)$. Fourth,

$$\sum_{n=\lfloor x\rfloor+1}^{\infty} (n+1+x)^{-s}(n-x)^{\sigma-N-1} = O\Big(x^{-\sigma}\big(\lfloor x\rfloor - x + 1\big)^{\sigma-N-1} + 1\Big).$$

And finally,

$$\begin{aligned}
\sum_{n=\lfloor x\rfloor+1}^{\infty} \Big((n+1+x)^{-s} - (n-x)^{-s}\Big) &= \sum_{n=\lfloor x\rfloor+1}^{\infty} n^{-\sigma} O\Big((1+\tfrac{1+x}{n})^{-\sigma} - (1-\tfrac{x}{n})^{-\sigma}\Big) \\
&= \sum_{n=\lfloor x\rfloor+1}^{\infty} \Big(O(n^{-\sigma-1}) + O(n^{-\sigma-2}) + \dots\Big) \\
&= O(1).
\end{aligned}$$

Substituting all of the above estimates into the expression for G_B gives:

$$\begin{aligned}
G_B = \frac{e^{-i\pi s/2}}{(2\pi)^s} &\sum_{n=0}^{\left\lfloor x-\frac{t}{\eta}\right\rfloor} \frac{\Gamma(s)}{(x-n)^s} \\
&- \frac{e^{i\pi\sigma/2}e^{it\log\eta}}{(2\pi)^s} \sum_{n=0}^{\infty}\sum_{j=0}^{N-1} e^{i(n-x)\eta}\left[\left(\tfrac{1}{n-x-\frac{it}{z}}\cdot\tfrac{\mathrm{d}}{\mathrm{d}z}\right)^j\left(\tfrac{z^{\sigma-1}}{n-x-\frac{it}{z}}\right)\right]_{z=-i\eta} \\
&+ \frac{e^{i\pi\sigma/2}e^{it\log\eta}}{(2\pi)^s} \sum_{n=0}^{\infty}\sum_{j=0}^{N-1} e^{i(x+n+1)\eta}\left[\left(\tfrac{1}{x+n+1-\frac{it}{z}}\cdot\tfrac{\mathrm{d}}{\mathrm{d}z}\right)^j\left(\tfrac{z^{\sigma-1}}{x+n+1-\frac{it}{z}}\right)\right]_{z=-i\eta} \\
&+ O\Big((2N+1)!!(N+1)^2\sigma^{-1}t^{\sigma-N-1}x^{1-\sigma}\Big) \\
&+ O\Big((2N+1)!!(N+1)\eta^{\sigma-N-1}(x-\lfloor x\rfloor)^{-N-1}\epsilon^{-2N-2}(1+\epsilon)^{2N+2}\Big) \\
&+ O\Big((2N+1)!!(N+1)^2\sigma^{-1}\eta^{\sigma-1}t^{-N}\epsilon^{-2N-2}\Big) \\
&+ O\Big((2N+1)!!(N+1)\eta^{\sigma-N-1}x^{-\sigma}\big(\lfloor x\rfloor - x+1\big)^{\sigma-N-1}\Big) \\
&+ O\Big((2N+1)!!(N+1)^2\sigma^{-1}t^{\sigma-N-1}\Big).
\end{aligned}$$

Each of the five error terms can be approximated by $(2N+1)!!(N+1)^2\sigma^{-1}$ times either $t^{\sigma-N-1}$ or $\eta^{\sigma-N-1}$ times one of $x^{1-\sigma}$, $(x-\lfloor x\rfloor)^{-N-1}\epsilon^{-2N-2}(1+\epsilon)^{2N+2}$, and $x^{-\sigma}\big(\lfloor x\rfloor - x + 1\big)^{\sigma-N-1}$. Thus, we finally get the required form of the error terms.

It remains to prove that the infinite series over n can be reduced to a finite one by finding an appropriate upper bound $M(N,x,\eta)$: in other words, to find M large enough that the series

$$\sum_{n=M+1}^{\infty}\sum_{j=0}^{N-1} e^{i(n-x)\eta}\left[\left(\frac{1}{n-x-\frac{it}{z}}\cdot\frac{\mathrm{d}}{\mathrm{d}z}\right)^{j}\left(\frac{z^{\sigma-1}}{n-x-\frac{it}{z}}\right)\right]_{z=-i\eta} \tag{211}$$

and

$$\sum_{n=M+1}^{\infty}\sum_{j=0}^{N-1} e^{i(x+n+1)\eta}\left[\left(\frac{1}{x+n+1-\frac{it}{z}}\cdot\frac{\mathrm{d}}{\mathrm{d}z}\right)^{j}\left(\frac{z^{\sigma-1}}{x+n+1-\frac{it}{z}}\right)\right]_{z=-i\eta} \tag{212}$$

can be swallowed up by the existing remainder term.

We assume

$$M > x, \tag{213}$$

so that both series above can be estimated using the definition of D_N and the result (180) from Lemma 3.1.10. The expressions (211) and (212) can be rewritten as

$$\begin{aligned}
&\sum_{n=M+1}^{\infty}\sum_{j=0}^{N-1}\frac{e^{i(n-x)\eta}}{n-x+\frac{t}{\eta}}D_j(-i\eta;n-x;\sigma,t)\\
&=\sum_{n=M+1}^{\infty}\frac{e^{i(n-x)\eta}(-i\eta)^{\sigma-1}}{n-x+\frac{t}{\eta}}+\sum_{n=M+1}^{\infty}\sum_{j=1}^{N-1}O\left(\frac{1}{n-x+\frac{t}{\eta}}(2j-1)!!(j+1)^2\eta^{\sigma-j-1}(n-x)^{-j}\right)\\
&=O\left(\eta^{\sigma-1}\sum_{n=M+1}^{\infty}\frac{e^{in\eta}}{n-x}\right)+\sum_{j=1}^{N-1}O\left((2j-1)!!(j+1)^2\eta^{\sigma-j-1}\sum_{n=M+1}^{\infty}(n-x)^{-j-1}\right)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=M+1}^{\infty}\sum_{j=0}^{N-1}\frac{e^{i(x+n+1)\eta}}{x+n+1+\frac{t}{\eta}}D_j(-i\eta;x+n+1;\sigma,t)\\
&=\sum_{n=M+1}^{\infty}\frac{e^{i(x+n+1)\eta}(-i\eta)^{\sigma-1}}{x+n+1+\frac{t}{\eta}}\\
&\qquad+\sum_{n=M+1}^{\infty}\sum_{j=1}^{N-1}O\left(\frac{1}{x+n+1+\frac{t}{\eta}}(2j-1)!!(j+1)^2\eta^{\sigma-j-1}(x+n+1)^{-j}\right)\\
&=O\left(\eta^{\sigma-1}\sum_{n=M+1}^{\infty}\frac{e^{in\eta}}{x+n+1}\right)+\sum_{j=1}^{N-1}O\left((2j-1)!!(j+1)^2\eta^{\sigma-j-1}\sum_{n=M+1}^{\infty}(x+n+1)^{-j-1}\right).
\end{aligned}$$

Both of these can be bounded by $O\big((2N-1)!!(N+1)^2\eta^{\sigma-N-1}\big)$ just as in Lemma 3.1.11, provided that M satisfies the conditions

$$\sum_{n=M+1}^{\infty} e^{in\eta}(n-x)^{-1}\le\eta^{-N} \tag{214}$$

and

$$\sum_{n=M+1}^{\infty} (n-x)^{-2} \le \eta^{1-N}. \tag{215}$$

Thus, for any M satisfying the conditions (213) and (214) and (215), the remainder term from the tail of the n-series is swallowed up by the other remainder terms, and we have the desired result. Note that if $\eta \in 2\pi\mathbb{Z}$ then the series in (214) is divergent and so we need $M = \infty$. □

3.1.6 Final result

Theorem 3.1.19. *The modified Hurwitz zeta function is given by the following finite asymptotic series:*

$$\begin{aligned}
\zeta_1(x,s) = {} & \sum_{n=1}^{\left\lfloor \frac{t}{\eta}-x \right\rfloor} \frac{1}{(x+n)^s} - \sum_{n=0}^{\left\lfloor x-\frac{t}{\eta} \right\rfloor} \frac{1}{(x-n)^s} + \chi(s)\Bigg(\sum_{m=1}^{\lfloor \eta/2\pi \rfloor} e^{-2\pi i m x} m^{s-1} \\
& + \frac{e^{-i\pi\sigma/2} e^{it\log\eta}}{(2\pi)^s} \sum_{n=1}^{M} \sum_{j=0}^{N-1} e^{-i(x+n)\eta} \left[\left(\tfrac{1}{x+n-\frac{it}{z}} \cdot \tfrac{\mathrm{d}}{\mathrm{d}z} \right)^j \left(\tfrac{z^{\sigma-1}}{x+n-\frac{it}{z}} \right) \right]_{z=i\eta} \\
& + \frac{e^{i\pi\sigma/2} e^{it\log\eta}}{(2\pi)^s} \sum_{n=0}^{M} \sum_{j=0}^{N-1} e^{i(n-x)\eta} \left[\left(\tfrac{1}{n-x-\frac{it}{z}} \cdot \tfrac{\mathrm{d}}{\mathrm{d}z} \right)^j \left(\tfrac{z^{\sigma-1}}{n-x-\frac{it}{z}} \right) \right]_{z=-i\eta} \\
& + O\Big((2N+1)!!(N+1)^2 \sigma^{-1} \min(t,\eta)^{\sigma-N-1} x^{-\sigma} K_N(x) \big(\tfrac{1+\epsilon}{\epsilon}\big)^{2N+2} \Big) \Bigg),
\end{aligned} \tag{216}$$

where $s = \sigma + it$, $0 < \sigma \le 1$, $0 < t < \infty$, $0 < x < \infty$, $0 < \eta < \infty$ satisfies Assumption 3.1.9 for some fixed $\epsilon > 0$, M is a natural number depending only on N, x, and η (or $M = \infty$ if $\eta \in 2\pi\mathbb{Z}$), the function $K_N(x)$ is defined by

$$K_N(x) = \max\Big(x, (x - \lfloor x \rfloor)^{-N-1}, (\lfloor x \rfloor - x + 1)^{-N-1} \Big),$$

and the O-constant is uniform in all variables.

Proof. Substituting the results of Lemma 3.1.11, Lemma 3.1.14, and Lemma 3.1.18 into the identity (170), we find:

$$\zeta_1(x,s) = \chi(s)\Bigg(\sum_{m=1}^{\lfloor \eta/2\pi \rfloor} e^{-2\pi i m x} m^{s-1} + \frac{e^{-i\pi s/2}}{(2\pi)^s} \sum_{n=1}^{\left\lfloor \frac{t}{\eta} - x \right\rfloor} \frac{\Gamma(s)}{(x+n)^s} - \frac{e^{-i\pi s/2}}{(2\pi)^s} \sum_{n=0}^{\left\lfloor x - \frac{t}{\eta} \right\rfloor} \frac{\Gamma(s)}{(x-n)^s}$$

$$+ \frac{e^{i\pi\sigma/2} e^{it\log\eta}}{(2\pi)^s} \sum_{n=1}^{M} \sum_{j=0}^{N-1} e^{i(x+n)\eta} \left[\left(\frac{1}{x+n-\frac{it}{z}} \cdot \frac{\mathrm{d}}{\mathrm{d}z} \right)^j \left(\frac{z^{\sigma-1}}{x+n-\frac{it}{z}} \right) \right]_{z=-i\eta}$$

$$+ \frac{e^{-i\pi\sigma/2} e^{it\log\eta}}{(2\pi)^s} \sum_{n=1}^{M} \sum_{j=0}^{N-1} e^{-i(x+n)\eta} \left[\left(\frac{1}{x+n-\frac{it}{z}} \cdot \frac{\mathrm{d}}{\mathrm{d}z} \right)^j \left(\frac{z^{\sigma-1}}{x+n-\frac{it}{z}} \right) \right]_{z=i\eta}$$

$$+ \frac{e^{i\pi\sigma/2} e^{it\log\eta}}{(2\pi)^s} \sum_{n=0}^{M} \sum_{j=0}^{N-1} e^{i(n-x)\eta} \left[\left(\frac{1}{n-x-\frac{it}{z}} \cdot \frac{\mathrm{d}}{\mathrm{d}z} \right)^j \left(\frac{z^{\sigma-1}}{n-x-\frac{it}{z}} \right) \right]_{z=-i\eta}$$

$$- \frac{e^{i\pi\sigma/2} e^{it\log\eta}}{(2\pi)^s} \sum_{n=0}^{M} \sum_{j=0}^{N-1} e^{i(x+n+1)\eta} \left[\left(\frac{1}{x+n+1-\frac{it}{z}} \cdot \frac{\mathrm{d}}{\mathrm{d}z} \right)^j \left(\frac{z^{\sigma-1}}{x+n+1-\frac{it}{z}} \right) \right]_{z=-i\eta}$$

$$+ O\Big((2N-1)!!(N+1)^2 \eta^{\sigma-N-1} \Big)$$

$$+ O\big((2N+1)!!(N+1)^2 \sigma^{-1} \eta^{\sigma-N-1} \epsilon^{-2N-2} (1+\epsilon)^{2N+2} \big)$$

$$+ O\Big((2N+1)!!(N+1)^2 \sigma^{-1} \min(t,\eta)^{\sigma-N-1} \big[x^{1-\sigma}$$

$$+ (x - \lfloor x \rfloor)^{-N-1} \big(\tfrac{1+\epsilon}{\epsilon} \big)^{2N+2} + x^{\sigma} \big(\lfloor x \rfloor - x + 1 \big)^{-N-1} \big] \Big) \Bigg)$$

$$= \chi(s)\Bigg(\sum_{m=1}^{\lfloor \eta/2\pi \rfloor} e^{-2\pi i m x} m^{s-1} + \frac{e^{-i\pi s/2}}{(2\pi)^s} \sum_{n=1}^{\left\lfloor \frac{t}{\eta} - x \right\rfloor} \frac{\Gamma(s)}{(x+n)^s} - \frac{e^{-i\pi s/2}}{(2\pi)^s} \sum_{n=0}^{\left\lfloor x - \frac{t}{\eta} \right\rfloor} \frac{\Gamma(s)}{(x-n)^s}$$

$$+ \frac{e^{-i\pi\sigma/2} e^{it\log\eta}}{(2\pi)^s} \sum_{n=1}^{M} \sum_{j=0}^{N-1} e^{-i(x+n)\eta} \left[\left(\frac{1}{x+n-\frac{it}{z}} \cdot \frac{\mathrm{d}}{\mathrm{d}z} \right)^j \left(\frac{z^{\sigma-1}}{x+n-\frac{it}{z}} \right) \right]_{z=i\eta}$$

$$+ \frac{e^{i\pi\sigma/2} e^{it\log\eta}}{(2\pi)^s} \sum_{n=0}^{M} \sum_{j=0}^{N-1} e^{i(n-x)\eta} \left[\left(\frac{1}{n-x-\frac{it}{z}} \cdot \frac{\mathrm{d}}{\mathrm{d}z} \right)^j \left(\frac{z^{\sigma-1}}{n-x-\frac{it}{z}} \right) \right]_{z=-i\eta}$$

$$+ O\Big((2N+1)!!(N+1)^2 \sigma^{-1} \min(t,\eta)^{\sigma-N-1} \big[x^{1-\sigma}$$

$$+ (x - \lfloor x \rfloor)^{-N-1} \big(\tfrac{1+\epsilon}{\epsilon} \big)^{2N+2} + x^{-\sigma} \big(\lfloor x \rfloor - x + 1 \big)^{-N-1} \big] \Big) \Bigg).$$

The second and third series in this expression can be simplified by using (19) together with Euler's reflection formula $\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)}$, $s \in \mathbb{C}\backslash\mathbb{Z}$, to obtain:

$$\chi(s) \tfrac{e^{-i\pi s/2}}{(2\pi)^s} \Gamma(s) = \tfrac{(2\pi)^s}{\pi} \Gamma(1-s) \sin\left(\tfrac{\pi s}{2}\right) \tfrac{e^{-i\pi s/2}}{(2\pi)^s} \Gamma(s) = \tfrac{e^{-i\pi s/2}}{\pi} \sin\left(\tfrac{\pi s}{2}\right) \tfrac{\pi}{\sin(\pi s)}$$

$$= \frac{e^{-i\pi s/2}}{2\cos\left(\frac{\pi s}{2}\right)} = \frac{e^{-i\pi s/2}}{e^{i\pi s/2} + e^{-i\pi s/2}} = \frac{1}{1 + e^{i\pi\sigma} e^{-\pi t}} = 1 + O\big(e^{-\pi t}\big).$$

Since exponential decay in t is negligible in the large-t asymptotics considered here, we can therefore rewrite the above formula for $\zeta_1(x,s)$ as follows:

$$\begin{aligned}\zeta_1(x,s) = &\sum_{n=1}^{\left\lfloor \frac{t}{\eta}-x\right\rfloor} \frac{1}{(x+n)^s} - \sum_{n=0}^{\left\lfloor x-\frac{t}{\eta}\right\rfloor} \frac{1}{(x-n)^s} + \chi(s)\Bigg(\sum_{m=1}^{\lfloor \eta/2\pi\rfloor} e^{-2\pi i m x} m^{s-1} \\ &+ \frac{e^{-i\pi\sigma/2}e^{it\log\eta}}{(2\pi)^s} \sum_{n=1}^{M}\sum_{j=0}^{N-1} e^{-i(x+n)\eta}\left[\left(\frac{1}{x+n-\frac{it}{z}}\cdot\frac{\mathrm{d}}{\mathrm{d}z}\right)^j\left(\frac{z^{\sigma-1}}{x+n-\frac{it}{z}}\right)\right]_{z=i\eta} \\ &+ \frac{e^{i\pi\sigma/2}e^{it\log\eta}}{(2\pi)^s} \sum_{n=0}^{M}\sum_{j=0}^{N-1} e^{i(n-x)\eta}\left[\left(\frac{1}{n-x-\frac{it}{z}}\cdot\frac{\mathrm{d}}{\mathrm{d}z}\right)^j\left(\frac{z^{\sigma-1}}{n-x-\frac{it}{z}}\right)\right]_{z=-i\eta} \\ &+ O\Big((2N+1)!!(N+1)^2\sigma^{-1}\min(t,\eta)^{\sigma-N-1}\big[x^{1-\sigma} \\ &+ (x-\lfloor x\rfloor)^{-N-1}\left(\tfrac{1+\epsilon}{\epsilon}\right)^{2N+2} + x^{-\sigma}\left(\lfloor x\rfloor - x+1\right)^{-N-1}\big]\Big)\Bigg).\end{aligned}$$

Then the required result (216) follows. The dependence of M on N, x, and η is given by the equations (213), (214), and (215), since the previous conditions (183) and (184) are implied by these. □

Note that (216) certainly describes a valid asymptotic series – each term in the series over j being smaller than the last, and the remainder term smaller than all of them – precisely because the result is valid for all N. The remainder term is of an order in t which decreases as N increases, and reducing N is equivalent to removing terms from the end of the series, so the estimate for the remainder term gives us the order of each term in the series over j.

Corollary 3.1.20. *With all notation and assumptions as in Theorem 3.1.19, the leading-order asymptotics for the Hurwitz zeta function can be expressed by the formulae below.*

Case 1: *if $\eta > \frac{t}{x}$, then*

$$\zeta_1(x,s) \sim -\sum_{n=0}^{\left\lfloor x-\frac{t}{\eta}\right\rfloor} \frac{1}{(x-n)^s} + \chi(s)\sum_{m=1}^{\lfloor \eta/2\pi\rfloor} e^{-2\pi i m x} m^{s-1}. \tag{217}$$

Case 2: *if $\eta < \frac{t}{x+1}$, then*

$$\zeta_1(x,s) \sim \sum_{n=1}^{\left\lfloor \frac{t}{\eta}-x\right\rfloor} \frac{1}{(x+n)^s} + \chi(s)\sum_{m=1}^{\lfloor \eta/2\pi\rfloor} e^{-2\pi i m x} m^{s-1}. \tag{218}$$

Case 3: *if* $\frac{t}{x+1} < \eta < \frac{t}{x}$*, then*

$$\zeta_1(x,s) \sim \chi(s) \sum_{m=1}^{\lfloor \eta/2\pi \rfloor} e^{-2\pi i m x} m^{s-1}. \tag{219}$$

Proof. If $\eta > \frac{t}{x}$, then $\frac{t}{\eta} - x < 0$, thus the first sum in (216) vanishes.

If $\eta < \frac{t}{x+1}$, then $x - \frac{t}{\eta} < -1$, thus the second sum in (216) vanishes.

If $\frac{t}{x+1} < \eta < \frac{t}{x}$, then $\frac{t}{\eta} - x < 1$ and $x - \frac{t}{\eta} < 0$, thus the first and second sums in (216) both vanish. □

By Assumption 3.1.9, η cannot be equal to either $\frac{t}{x}$ or $\frac{t}{x+1}$. Thus, the three cases above cover *all* possibilities for η.

Remark 3.1.21. Let us consider the particular value $x = 0$, and compare the results of Theorem 3.1.19 with the results for $\zeta(s)$ obtained in [57].

The three cases considered in the above corollary correspond to the cases into which the problem was separated in [57]. Case 1 above is not possible when $x = 0$, but Case 2 above now becomes the $\eta < t$ case of [57], and Case 3 above becomes the $\eta > t$ case of [57]. The case $\eta = t$, which is considered in Theorem 3.2 of [57], is prohibited when $x = 0$, under the terms of Assumption 3.1.9.

When $\eta < t$, the formulae for $\zeta(s)$ obtained in Theorems 4.1 and 4.4 of [57] were derived by entirely different methods from those used here, so it would be difficult to compare them with the result of our Theorem 3.1.19 without reducing the relevant expressions all the way back to the original form of $\zeta_1(x,s)$. However, we can easily check that the leading-order terms of the expressions derived in [57] are identical with those of Theorem 3.1.19: both the expressions proven in Theorems 4.1 and 4.4 of [57] yield

$$\zeta(s) \sim \sum_{n=1}^{\lfloor \frac{t}{\eta} \rfloor} \frac{1}{n^s} + \chi(s) \sum_{m=1}^{\lfloor \eta/2\pi \rfloor} m^{s-1},$$

and this is precisely the expression obtained from (218) under the assumption that $x = 0$.

When $\eta > (1+\epsilon)t$ for some $\epsilon > 0$, the formula for $\zeta(s)$ obtained in Theorem 3.1 of [57] can be written as follows:

$$\begin{aligned}\zeta(1-s) = \sum_{n=1}^{\lfloor \frac{\eta}{2\pi} \rfloor} n^{s-1} - \frac{1}{s}\left(\frac{\eta}{2\pi}\right)^s + \frac{e^{i\pi s/2}}{(2\pi)^s} \sum_{n=1}^{\infty} \sum_{j=0}^{N-1} e^{-nz+it\log z} \left[\left(\frac{1}{n-\frac{it}{z}} \cdot \frac{\mathrm{d}}{\mathrm{d}z}\right)^j \left(\frac{z^{\sigma-1}}{n-\frac{it}{z}}\right)\right]_{z=-i\eta} \\ + \frac{e^{-i\pi s/2}}{(2\pi)^s} \sum_{n=1}^{\infty} \sum_{j=0}^{N-1} e^{-nz+it\log z} \left[\left(\frac{1}{n-\frac{it}{z}} \cdot \frac{\mathrm{d}}{\mathrm{d}z}\right)^j \left(\frac{z^{\sigma-1}}{n-\frac{it}{z}}\right)\right]_{z=i\eta} \\ + O\Big((2N+1)!!N\left(\tfrac{1+\epsilon}{\epsilon}\right)^{2(N+1)} \eta^{\sigma-N-1}\Big).\end{aligned} \tag{220}$$

On the other hand, our result (216), with $x=0$ and $\eta>t$, becomes:

$$\zeta(s)=\chi(s)\Bigg(\sum_{m=1}^{\lfloor\eta/2\pi\rfloor}m^{s-1}+\frac{e^{-i\pi\sigma/2}e^{it\log\eta}}{(2\pi)^s}\sum_{n=1}^{M}\sum_{j=0}^{N-1}e^{-in\eta}\left[\left(\tfrac{1}{n-\frac{it}{z}}\cdot\tfrac{\mathrm{d}}{\mathrm{d}z}\right)^j\left(\tfrac{z^{\sigma-1}}{n-\frac{it}{z}}\right)\right]_{z=i\eta}$$
$$+\frac{e^{i\pi\sigma/2}e^{it\log\eta}}{(2\pi)^s}\sum_{n=0}^{M}\sum_{j=0}^{N-1}e^{in\eta}\left[\left(\tfrac{1}{n-\frac{it}{z}}\cdot\tfrac{\mathrm{d}}{\mathrm{d}z}\right)^j\left(\tfrac{z^{\sigma-1}}{n-\frac{it}{z}}\right)\right]_{z=-i\eta}$$
$$+O\Big((2N+1)!!(N+1)^2t^{\sigma-N-1}\left(\tfrac{1+\epsilon}{\epsilon}\right)^{2N+2}\Big)\Bigg). \tag{221}$$

Note that our assumption $\eta>(1+\epsilon)t$ guarantees that Assumption 3.1.9 is valid, because $1>(1+\epsilon)\frac{t}{\eta}$ and $0<(1-\epsilon)\frac{t}{\eta}$. Using the well-known identity $\zeta(s)=\chi(s)\zeta(1-s)$, as well as our already-known bounds on series tails which enable the infinite sums over n in (220) to be replaced by finite sums up to M, it is straightforward to check that equations (220) and (221) are equivalent. In particular, the term

$$-\frac{1}{s}\left(\frac{\eta}{2\pi}\right)^s$$

in (220) comes from the $n=0$ part of the second series in (221):

$$\begin{aligned}
&\frac{e^{i\pi\sigma/2}e^{it\log\eta}}{(2\pi)^s}\sum_{j=0}^{N-1}\left[\left(\tfrac{1}{-\frac{it}{z}}\cdot\tfrac{\mathrm{d}}{\mathrm{d}z}\right)^j\left(\tfrac{z^{\sigma-1}}{-\frac{it}{z}}\right)\right]_{z=-i\eta}\\
&=\frac{e^{i\pi\sigma/2}\eta^{it}}{(2\pi)^s}\sum_{j=0}^{N-1}\left[(-it)^{-j-1}\left(z\cdot\tfrac{\mathrm{d}}{\mathrm{d}z}\right)^j\left(z^\sigma\right)\right]_{z=-i\eta}=\frac{e^{i\pi\sigma/2}\eta^{it}}{(2\pi)^s}\sum_{j=0}^{N-1}\left[(-it)^{-j-1}\sigma^jz^\sigma\right]_{z=-i\eta}\\
&=\frac{e^{i\pi\sigma/2}\eta^{it}}{(2\pi)^s}\sum_{j=0}^{N-1}\frac{1}{-it}\left(\frac{\sigma}{-it}\right)^je^{-i\pi\sigma/2}\eta^\sigma=\frac{\eta^s}{(2\pi)^s}\cdot\frac{1}{-it}\cdot\frac{1-\left(\frac{\sigma}{-it}\right)^N}{1-\frac{\sigma}{-it}}\\
&=-\frac{1}{s}\left(\frac{\eta}{2\pi}\right)^s\left(1-\left(\tfrac{\sigma}{-it}\right)^N\right),
\end{aligned}$$

and the t^{-N} part is absorbed by the error term.

Thus, we have shown that the results established here are consistent, as expected, with the existing results of [57] for the Riemann zeta function. □

3.2 Asymptotics in the neighbourhood of a stationary point

3.2.1 Introduction

In this chapter, we consider the integral expression J_B defined by Definition 3.2.1, and derive its large-t asymptotics. The significance of this expression derives from its appearance in important problems relating to the Lindelöf hypothesis [56], which motivated the necessary work contained in this chapter. The problem is to find asymptotics for J_B as $t \to \infty$, for λ in the range (222), with the error term independent of λ.

Definition 3.2.1 (The integral J_B)**.** Let δ and σ be fixed constants satisfying $0 < \delta < 1$ and $\frac{1}{2} \leq \sigma < 1$, and let λ satisfy

$$\frac{t^{\delta-1}}{1-t^{\delta-1}} \leq \lambda \leq t^{1-\delta} - 1. \tag{222}$$

Let

$$J_B(t; \lambda, \delta, \sigma) \coloneqq \int_{1-t^{\delta-1}}^{\infty e^{i\phi}} (1-z)^{-1/2} z^{\sigma-1/2} \exp\left(itF(z;\lambda)\right) \mathrm{d}z, \tag{223}$$

where the function F is defined by

$$F(z;\lambda) \coloneqq (1-z)\log(1-z) + z\log z + z\log\lambda, \tag{224}$$

with its branch cuts (as a function of z) along $(-\infty, 0] \cup [1, \infty)$, and the angle ϕ satisfies

$$0 < \phi < \pi/2 \quad \text{if } \log\lambda \geq 0, \qquad 0 < \phi < \arctan\left(\frac{\pi}{\left|\log\lambda\right|}\right) \quad \text{if } \log\lambda < 0. \tag{225}$$

In general, it is known that the main contributions to the large-t asymptotics of an integral of the form

$$\int_{\alpha}^{\beta} g(x) e^{ith(x)} \, \mathrm{d}x,$$

where the functions $g(x)$ and $h(x)$ are sufficiently smooth, come from the neighbourhood of the endpoints α and β, and from the neighbourhood of *stationary points* of $h(x)$, i.e. points where $h'(x) = 0$.

The case when the stationary point is close to an endpoint was considered by Bleistein [27], who introduced a general algorithm for finding uniform asymptotics using a global change of variables and integration by parts. This methodology is described in [145, Chapter VII]. In the case of our integral J_B, in §2–3 of [52] this type of argument was successfully applied to derive rigorous uniform error estimates, the final result being Theorem 3.2.3 below.

But this method, as well as requiring a lot of substitutions and complicated rewriting of the problem, yields only the leading-order asymptotics of J_B. It turns out that it is possible to use a more direct argument, without needing a global change of variables, in order to obtain asymptotics to *all* orders of the integral J_B in the main case of $\sigma = \frac{1}{2}$. This was my contribution to [52], and also the main content of the current chapter.

Remark 3.2.2. It is straightforward to check that the requirements (225) are equivalent to demanding that $\mathrm{Im}F(z;\lambda) > 0$ for large $|z|$, i.e. that the integrand in (223) has exponential decay for large $|z|$.

Since we are not interested in the dependence of the error term on the parameters σ and δ, we will suppress the dependence of J_B (and all other functions) on these two variables; i.e. we write $J_B = J_B(t;\lambda)$ only.

An important question to ask is: **why is this problem difficult to solve?** Since

$$\frac{\partial F}{\partial z}(z,\lambda) = \log\left(\frac{z\lambda}{1-z}\right), \tag{226}$$

there is a stationary point at $z = \frac{1}{1+\lambda}$. When $\frac{t^{\delta-1}}{1-t^{\delta-1}} < \lambda \le t^{1-\delta} - 1$, the stationary point is in the interval $(0, 1-t^{\delta-1})$, and thus is away from the contour of integration. However, when λ equals its lower bound λ_c, defined by

$$\lambda_c \coloneqq \frac{t^{\delta-1}}{1-t^{\delta-1}} = \frac{1}{t^{1-\delta}-1}, \tag{227}$$

then the stationary point is at $z = 1 - t^{\delta-1}$, i.e. at the endpoint of integration.

The difficulty lies in finding a *uniform* expression for the asymptotics of J_B, which is valid regardless of the value of λ in the range (222), i.e. regardless of how close the stationary point is to the contour of integration. Somehow we need to make a continuous transition between the asymptotics resulting from the stationary phase method (stationary point on the contour) and those resulting from integration by parts (stationary point away from the contour).

The result on uniform leading-order asymptotics for J_B which comes from adapting the method of Bleistein and performing a global change of variables is as follows.

Theorem 3.2.3 (Uniform leading-order asymptotics of J_B)**.** *We introduce new variables $Z = Z(z)$ and $\Lambda = \Lambda(\lambda)$ so that $Z = 0$ corresponds to $z = 1-t^{\delta-1}$ and $\Lambda = 0$ corresponds to $\lambda = \lambda_c$; i.e. we let*

$$\lambda = \lambda_c(1+\Lambda) \quad \text{and} \quad z = \frac{(1+\lambda_c Z)}{1+\lambda_c}. \tag{228}$$

Let

$$\omega(t,\Lambda) \coloneqq \sqrt{\frac{\lambda_c t}{2}\frac{\log(1+\Lambda)}{1+\lambda_c}}, \tag{229}$$

and observe that $\omega \geq 0$ since $\Lambda \geq 0$.

The leading-order asymptotics of J_B are given by

$$J_B(t;\lambda) = \left(\frac{\lambda_c}{1+\lambda_c}\right)^{1/2} \left(\frac{1}{1+\lambda_c}\right)^{\sigma-1/2} \exp\left(\frac{itf_0(\Lambda,\lambda_c)}{(1+\lambda_c)}\right) \tilde{J}_B(t;\lambda), \tag{230}$$

where f_0 is defined by

$$f_0(\Lambda,\lambda_c) \coloneqq \log\left(\frac{\lambda_c(1+\Lambda)}{1+\lambda_c}\right) - \lambda_c \log\left(\frac{1+\lambda_c}{\lambda_c}\right), \tag{231}$$

and the leading-order asymptotics of $\tilde{J}_B$ are given by

$$\tilde{J}_B(t;\lambda) = e^{-i\omega^2}\sqrt{\frac{2}{\lambda_c t}}\left(\int_\omega^{\infty e^{i\pi/4}} e^{i\xi^2}\mathrm{d}\xi\right)\Big(1+o(1)\Big), \tag{232}$$

where the $o(1)$ is independent of Λ. The branch cut for $\log(1+\lambda_c Z)$ is taken from $-1/\lambda_c$ to ∞ on the negative real axis and the branch cut for $\log(1-Z)$ from 1 to ∞ is taken on the positive real axis. Note that the range for λ in (222) means the parameter Λ satisfies

$$0 \leq \Lambda < \frac{t^{1-\delta}-1}{\lambda_c} - 1. \tag{233}$$

If Λ is such that $\omega = O(1)$ as $t \to \infty$ (e.g. $\Lambda = 0$), then the integral on the right-hand side of (232) is an $O(1)$ quantity independent of Λ. Furthermore, if Λ is such that $\omega \to \infty$ as $t \to \infty$, then

$$\tilde{J}_B(t;\lambda) = \sqrt{\frac{2}{\lambda_c t}}\left(\frac{-1}{2i\omega} + O\left(\frac{1}{\omega^3}\right)\right)\Big(1+o(1)\Big), \tag{234}$$

as $t \to \infty$, where both the $o(1)$ and the O-constant are independent of Λ.

Proof. See §2–3 of [52]. □

Remark 3.2.4. In the asymptotics (232), $\lambda_c t$ plays the role of the large parameter (recall from (227) that $\lambda_c \sim t^{\delta-1}$ as $t \to \infty$, so $\lambda_c t \sim t^\delta \to \infty$ as $t \to \infty$).

When Λ is such that $\omega = O(1)$ as $t \to \infty$, the right-hand side of (232) is $O((\lambda_c t)^{-1/2})$, i.e., the asymptotics expected from a stationary point. When Λ is such that $\omega \to \infty$ as $t \to \infty$, (234) implies that the right-hand side of (232) is $O((\lambda_c t)^{-1})$, i.e., the asymptotics expected from integration by parts away from a stationary point.

Thus, a reasonable transition is achieved between the "stationary point" and "integration by parts" contributions.

When one obtains the formal asymptotics of standard stationary-phase-type integrals (i.e. those without the stationary point approaching an endpoint), one splits the integral, uses local expansions near the stationary point, and then uses integration by parts away from the stationary point (see, e.g., [24, §6.5]). For the rigorous justification of these asymptotics, however, the standard approach is to first make a global change of variables in the same spirit as the method of Bleistein; see, e.g., [43, §2.4, §2.9], [145, Chapter 2 §1, §3], and [103, §3.3, §5.3]. It is rare to see examples in the literature where rigorous asymptotics are obtained *without* first making a global change of variables, but via directly splitting the integral and using local expansions and integration by parts; one notable exception is [24, §6.4, Examples 7 and 8].

It is therefore a challenging question whether the rigorous *uniform* leading-order asymptotics of J_B can be obtained *without* first making a global change of variables. The point of the current chapter is to show that this is indeed possible; in fact, we go even further by obtaining the asymptotics to *all* orders of J_B, in the most important case when $\sigma = \frac{1}{2}$.

Theorem 3.2.5 (Asymptotics of J_B to all orders when $\sigma = 1/2$). *In the case $\sigma = 1/2$, the asymptotics of J_B to all orders are given by*

$$\begin{aligned} J_B(t;\lambda) = &\sum_{j=1}^{m-3} T_j(t;\lambda) + O\left((4m-5)!!t^{-\frac{1}{2}-\frac{(4m-5)\delta}{2}}(\log t)^{(4m-3)/2}a^{-4m+4}\right) \\ &+ \exp\left(itF(1-t^{\delta-1};\lambda) - i\omega^2\right)t^{-1/2}\sqrt{\frac{2}{1+\lambda_c}}\left(\int_{\omega}^{\omega+a\sqrt{\lambda_c t/2}} e^{i\xi^2}\,\mathrm{d}\xi\right. \\ &\left. + O\left(t^{-\frac{1}{2}+\frac{3\delta}{2}}a^4\right)\right), \end{aligned} \tag{235}$$

for any natural number $m \geq 4$, where $a \gg t^{-\delta/2}$ is such that

$$t^{\epsilon-\frac{\delta}{2}+\frac{\delta}{4m+2}} \ll a \ll t^{-\frac{\delta}{2}+\frac{\delta}{4m-2}} \tag{236}$$

for some $\epsilon > 0$, and the summands T_j are defined exactly by (251) *below and can be estimated by* (266). *This expansion is uniform in the sense that the big-O terms are independent of λ.*

The main idea is to split the integral J_B into two parts: J_{B1}, an integral along a finite contour that is both real and (when t is large) vanishingly small, and J_{B2}, an integral along an infinite contour that is controllably "far" from the endpoint (and hence from the stationary point). The large-t asymptotics of J_{B2} can be computed to all orders, while those of J_{B1} can be computed at least to first order. This will be sufficient to find the asymptotics of J_B to all orders, since the error term in the asymptotic expression for J_{B1} can be made arbitrarily small compared to the asymptotics of J_{B2} by making an appropriate choice of the splitting point for the integrals.

The method can be summarised as follows:

- Step 1: split the contour of integration into a small part and an infinitely long part.
- Step 2: estimate the infinite-contour integral J_{B2} using integration by parts.
- Step 3: estimate the small-contour integral J_{B1} using a truncated Taylor series (i.e. a local expansion).
- Step 4: choose the splitting point for the contour so as to appropriately bound the remainder terms from both the previous steps.

3.2.2 Step 1: Splitting the integral

We define J_{B1} and J_{B2} as follows:

$$J_{B1}(t;\lambda) := \int_{1-t^{\delta-1}}^{1-k} (1-z)^{-1/2} e^{itF(z;\lambda)}\,\mathrm{d}z; \tag{237}$$

$$J_{B2}(t;\lambda) := \int_{1-k}^{\infty e^{i\phi}} (1-z)^{-1/2} e^{itF(z;\lambda)}\,\mathrm{d}z, \tag{238}$$

where $k = k(t,\delta)$ is chosen so that

$$0 < k < t^{\delta-1}. \tag{239}$$

We will fix k as a specific function of t and δ in Step 4. By Cauchy's theorem, we have

$$J_B(t;\lambda) = J_{B1}(t;\lambda) + J_{B2}(t;\lambda). \tag{240}$$

3.2.3 Step 2: The asymptotics of J_{B2}

We now prove two lemmas about the behaviour of the phase function $F(z;\lambda)$.

Lemma 3.2.6. *When z is on the contour*

$$\{z = 1-k+Re^{i\phi} : 0 \leq R < \infty\}, \tag{241}$$

then we have

$$\left|\frac{\partial F}{\partial z}\right| > \min\left(\frac{\pi}{2}-\phi, \log\left(\frac{t^{\delta-1}}{k}\right)\right), \tag{242}$$

and for sufficiently large R, $\left|\frac{\partial F}{\partial z}\right| > \frac{\pi}{2}-\phi$.

Proof. We split $\frac{\partial F}{\partial z}$ into its real and imaginary parts, as follows:

$$\frac{\partial F}{\partial z} = \log\left(\frac{z}{1-z}\right) + \log\lambda = \log\left|\frac{z}{1-z}\right| + \log\lambda + i\arg\left(\frac{z}{1-z}\right). \tag{243}$$

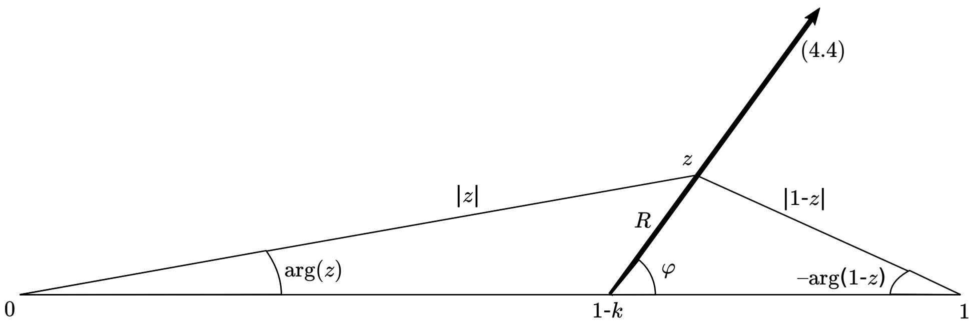

Figure 5: The geometry of the integration contour (241), for small R

As z moves along the contour (241) with R increasing, $\arg(z)$ is strictly increasing from 0 towards ϕ, and $\arg(1-z)$ is strictly decreasing from 0 towards $\phi - \pi$, so the imaginary part $\arg\left(\frac{z}{1-z}\right)$ increases monotonically from 0 towards π.

For very small $R > 0$, it is clear that $|z|$ is increasing and $|1-z|$ is decreasing (see Figure 5), so $\left|\frac{z}{1-z}\right|$ is increasing from its initial value of $\frac{1-k}{k} \sim k^{-1} \gg 1$ (t is large and $\delta > 0$ is small, and thus $k < t^{\delta-1}$ is small). But for very large R, it is clear that $\left|\frac{z}{1-z}\right| \sim 1$. So at some point the function $\left|\frac{z}{1-z}\right|$ must stop increasing and start decreasing, i.e. its derivative must change sign. We now find out where this point is by considering the simpler function $\left|\frac{z}{1-z}\right|^2$.

Using the parametrisation in (241), we write all the relevant functions in terms of R and not z:

$$\begin{aligned} z &= 1-k+R\cos\phi+iR\sin\phi; & 1-z &= k-R\cos\phi-iR\sin\phi; \\ |z|^2 &= (1-k)^2+2(1-k)R\cos\phi+R^2; & |1-z|^2 &= k^2-2kR\cos\phi+R^2. \end{aligned} \tag{244}$$

Now we can compute the value of R at which the derivative is zero:

$$\begin{aligned} \frac{\partial}{\partial R}\left(\left|\frac{z}{1-z}\right|^2\right) = 0 &\Leftrightarrow \left|1-z\right|^2\frac{\partial}{\partial R}\left(|z|^2\right) - |z|^2\frac{\partial}{\partial R}\left(\left|1-z\right|^2\right) = 0 \\ &\Leftrightarrow \left(k^2-2kR\cos\phi+R^2\right)\left(2(1-k)\cos\phi+2R\right) \\ &\qquad - \left((1-k)^2+2(1-k)R\cos\phi+R^2\right)\left(-2k\cos\phi+2R\right) = 0 \\ &\Leftrightarrow \big[-\cos\phi\big]R^2+\big[2k-1\big]R+\big[k(1-k)\cos\phi\big] = 0 \\ &\Leftrightarrow R = \frac{1-2k\pm\sqrt{1-4k\sin^2\phi+4k^2\sin^2\phi}}{-2\cos\phi}. \end{aligned}$$

Since k is small, we find the following first-order approximation for the critical value of R:

$$R = \frac{1-2k\pm(1-4k\sin^2\phi+4k^2\sin^2\phi)^{1/2}}{-2\cos\phi} \approx \frac{1-2k\pm(1-2k\sin^2\phi)}{-2\cos\phi}.$$

Taking the positive sign gives a negative value of R, so we take the negative sign and

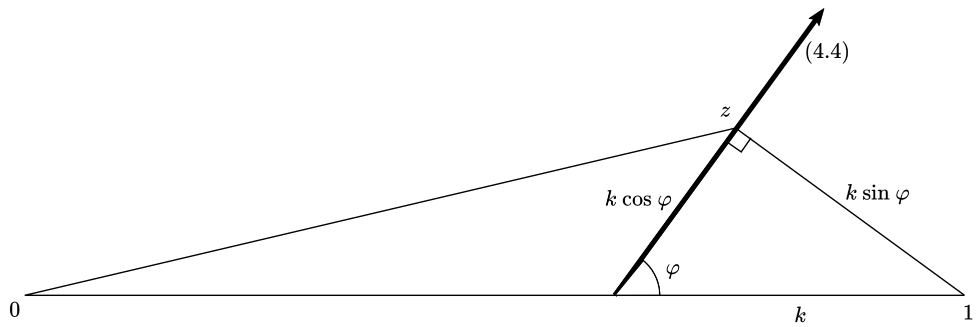

Figure 6: The geometry of the integration contour in (241) in the case when $R = k\cos\phi$

obtain

$$R \approx \frac{1 - 2k - 1 + 2k\sin^2\phi}{-2\cos\phi} = k\cos\phi.$$

Thus, we have proved that the function $\left|\frac{z}{1-z}\right|$ has a single stationary point for $R \geq 0$, namely a maximum at a value of R somewhere close to $k\cos\phi$. We now estimate the maximal value of $\left|\frac{z}{1-z}\right|$ by evaluating this function at $R = k\cos\phi$ precisely. With this value of R, using the above formulae, we have:

$$\begin{aligned}
|z|^2 &= (1-k)^2 + 2(1-k)k\cos^2\phi + k^2\cos^2\phi = 1 - 2k\sin^2\phi + k^2\sin^2\phi,\\
\left|1-z\right|^2 &= k^2 - 2k^2\cos^2\phi + k^2\cos^2\phi = k^2\sin^2\phi,\\
\left|\frac{z}{1-z}\right|^2 &= \frac{1}{k^2\sin^2\phi} - \frac{2}{k} + 1 \sim \frac{1}{k^2\sin^2\phi},
\end{aligned}$$

and thus

$$\left|\frac{z}{1-z}\right| \sim \frac{1}{k\sin\phi} > k^{-1}.$$

We are now in a position to estimate $\frac{\partial F}{\partial z}$, by bounding either its real part or its imaginary part according to the value of R. We split into two cases as follows.

Case 1: $\boldsymbol{R \geq k\cos\phi}$. Here we consider the imaginary part, namely $\arg\left(\frac{z}{1-z}\right)$. This is monotonically increasing and therefore bounded below by its value at $R = k\cos\phi$. We can see from Figure 6 that $R = k\cos\phi$ gives $\arg(1-z) = -(\frac{\pi}{2} - \phi)$ and therefore $\arg\left(\frac{z}{1-z}\right) > \pi/2 - \phi$. So in this case we have:

$$\left|\frac{\partial F}{\partial z}\right| > \frac{\pi}{2} - \phi. \tag{245}$$

Case 2: $\boldsymbol{R \leq k\cos\phi}$. Here we consider the real part, namely $\log\left|\frac{z}{1-z}\right| + \log\lambda$. We know that this quantity is monotonically increasing up to *approximately* $R = k\cos\phi$, and that its value at *precisely* $R = k\cos\phi$ is greater than its initial value, so it must be bounded below by its initial value, i.e. by $\log\left(\frac{(1-k)\lambda}{k}\right)$. Using the lower bound on λ in

(222), as well as the assumption $k < t^{\delta-1}$ from (239), we then have

$$\left|\frac{\partial F}{\partial z}\right| \geq \left|\log\left(\frac{1-k}{k}\right) + \log\lambda\right| > \left|\log\left(\frac{1}{k}-1\right) - \log\left(t^{1-\delta}-1\right)\right|$$
$$= \log\left(\frac{t^{\delta-1}}{k}\right) + \log(1-k) + \log\left(1-t^{\delta-1}\right) > \log\left(\frac{t^{\delta-1}}{k}\right). \tag{246}$$

Putting together the estimates (245) and (246), we have the desired result (242). And since (245) is valid for large R, we also have the second half of the claim. □

Motivated by the results of Lemma 3.2.6, we introduce the following notation, which will make some of the later calculations simpler.

Definition 3.2.7. Let

$$D = D(\lambda) \coloneqq \frac{\partial F}{\partial z}(1-k;\lambda) = \log\left(\frac{(1-k)\lambda}{k}\right), \tag{247}$$

and let

$$D_- \coloneqq \log\left(\frac{t^{\delta-1}}{k}\right).$$

Remark 3.2.8. We saw in (246) that $D(\lambda)$ always has D_- as a lower bound, but it can only be close to this value if λ is close to its critical value λ_c. In general D might be as large as $O(\log t)$.

As a corollary of Lemma 3.2.6, we can identify a particular situation where D_- is a lower bound for $\left|\frac{\partial F}{\partial z}\right|$ on the whole of the contour (241) (i.e. not just at the endpoint $z = 1-k$).

Corollary 3.2.9. *Suppose that k is close enough to $t^{\delta-1}$ that $D_- \ll 1$, and define an angle ϕ as follows:*

$$\phi = \frac{\pi}{4} \text{ if } \log\lambda \geq 0, \qquad \phi = \frac{1}{2}\arctan\left(\frac{\pi}{|\log\lambda|}\right) \text{ if } \log\lambda < 0, \tag{248}$$

noting that these choices satisfy the required conditions (225) on ϕ.

Then we have

$$\left|\frac{\partial F}{\partial z}(z;\lambda)\right| \geq D_-$$

for all z on the contour (241).

Proof. The definition (248) implies that $\phi \leq \frac{\pi}{4}$ regardless of λ, so $\frac{\pi}{2} - \phi \geq \frac{\pi}{4}$. We also have $D_- \ll 1$, so the result of Lemma 3.2.6 becomes

$$\left|\frac{\partial F}{\partial z}(z,\lambda)\right| > \min\left(\frac{\pi}{2} - \phi, D_-\right) = D_-,$$

as required. $\square$

Lemma 3.2.10. *When z is on the contour* (241)*, the function $F(z;\lambda)$ always has non-negative imaginary part.*

Proof. Clearly $\mathrm{Im}(F) = 0$ when $R = 0$, since then z and λ are both real.

When $R > 0$ is very small, we can estimate F as follows:

$$\begin{aligned}
F(z;\lambda) &= (k - Re^{i\phi})\log(k - Re^{i\phi}) \\
&\qquad + (1-k+Re^{i\phi})\log(1-k+Re^{i\phi}) + (1-k+Re^{i\phi})\log\lambda \\
&= (k - Re^{i\phi})\left(\log k + \log\left(1 - \frac{Re^{i\phi}}{k}\right)\right) \\
&\qquad + (1-k+Re^{i\phi})\left(\log(1-k) + \log\left(1 + \frac{Re^{i\phi}}{1-k}\right)\right) + (1-k+Re^{i\phi})\log\lambda \\
&\sim (k - Re^{i\phi})\left(\log k - \frac{Re^{i\phi}}{k}\right) \\
&\qquad + (1-k+Re^{i\phi})\left(\log(1-k) + \frac{Re^{i\phi}}{1-k}\right) + (1-k+Re^{i\phi})\log\lambda \\
&\sim \big[k\log k + (1-k)\log(1-k) + (1-k)\log\lambda\big] \\
&\qquad + Re^{i\phi}\big[-\log k - 1 + \log(1-k) + 1 + \log\lambda\big].
\end{aligned}$$

Therefore,

$$\mathrm{Im}(F) \sim R\sin\phi\Big[-\log k + \log(1-k) + \log\lambda\Big] = R\sin\phi\log\left(\lambda\left(\frac{1}{k} - 1\right)\right).$$

Thus, since $k < t^{\delta-1}$ and $\lambda \geq \frac{t^{\delta-1}}{1-t^{\delta-1}}$, we have $\mathrm{Im}(F) > 0$ for $R > 0$ small.

Clearly, $F(z;\lambda)$ is an analytic function of z for $\mathrm{Im}(z) > 0$, so

$$\begin{aligned}
\frac{\partial}{\partial R}\Big(\mathrm{Im}\, F\left(1-k+Re^{i\phi},\lambda\right)\Big) &= \frac{\partial}{\partial z}\Big(\mathrm{Im}\, F(z;\lambda)\Big)\Big|_{z=1-k+Re^{i\phi}} = \mathrm{Im}\left(\frac{\partial F}{\partial z}\right)\Big|_{z=1-k+Re^{i\phi}} \\
&= \Big(\arg(z) - \arg(1-z)\Big)\Big|_{z=1-k+Re^{i\phi}} > 0.
\end{aligned}$$

Thus, $\mathrm{Im}(F)$ is strictly increasing along the contour, which means it must be positive for all $R > 0$, as required. $\square$

We now prove the following lemma which allows us to simplify the terms arising from repeated integration by parts.

Lemma 3.2.11. *For any $N \in \mathbb{N}$,*

$$\left(\frac{\partial}{\partial z} \cdot \frac{-1}{it\frac{\partial F}{\partial z}}\right)^N \left((1-z)^{-1/2}\right) = \frac{(1-z)^{-(2N+1)/2}}{(-it)^N \left(\partial F/\partial z\right)^N} \sum_{m,n=0}^{N} A_{mn} z^{-m} \left(\frac{\partial F}{\partial z}\right)^{-n}, \qquad (249)$$

where F is defined by (224) *and the A_{mn} are dyadic rationals satisfying $|A_{mn}| < (3N)!$ for all m, n and $A_{mN} = 0$ for all $m < N$ and $A_{NN} = (-1)^N (2N-1)!!$, where we use the notation* $(2N-1)!! := (2N-1)(2N-3)\dots(5)(3)(1)$.

Proof. Firstly, the expression for $\partial F/\partial z$, (243), implies that

$$\frac{\partial^2 F}{\partial z^2} = \frac{1}{z} + \frac{1}{1-z} = \frac{1}{z(1-z)}.$$

So for $N = 1$ we have

$$\begin{aligned}\frac{\partial}{\partial z}\left(\frac{(1-z)^{-1/2}}{-it\frac{\partial F}{\partial z}}\right) &= \frac{(1-z)^{-3/2}\left(\frac{1}{2}\cdot\frac{\partial F}{\partial z} + (z-1)\frac{\partial^2 F}{\partial z^2}\right)}{-it\left(\partial F/\partial z\right)^2} \\ &= \frac{(1-z)^{-3/2}}{-it\left(\partial F/\partial z\right)}\left[\frac{1}{2} - z^{-1}\left(\frac{\partial F}{\partial z}\right)^{-1}\right],\end{aligned}$$

which is in the required form. For $N = 2$, similar calculations give that

$$\begin{aligned}&\frac{\partial}{\partial z}\left(\frac{(1-z)^{-3/2}\left(\frac{1}{2}\cdot\frac{\partial F}{\partial z} - \frac{1}{z}\right)}{-t^2\left(\partial F/\partial z\right)^3}\right) \\ &\qquad = \frac{(1-z)^{-5/2}}{-t^2\left(\partial F/\partial z\right)^2}\left[\frac{3}{4} - \left(\frac{7}{2}z^{-1} - z^{-2}\right)\left(\frac{\partial F}{\partial z}\right)^{-1} + 3z^{-2}\left(\frac{\partial F}{\partial z}\right)^{-2}\right],\end{aligned}$$

which is again in the required form.

For the general case, we proceed by induction. Assuming the equation (249) is valid with N replaced by $N-1$, and differentiating as before, we find that the LHS of (249) is:

$$\begin{aligned}\left(\frac{\partial}{\partial z}\cdot\frac{-1}{it\frac{\partial F}{\partial z}}\right)^N \left((1-z)^{-1/2}\right) &= \frac{\partial}{\partial z}\left[\frac{(1-z)^{-(2N-1)/2}}{(-it)^N\left(\partial F/\partial z\right)^N}\sum_{m,n=0}^{N-1} A_{mn} z^{-m}\left(\frac{\partial F}{\partial z}\right)^{-n}\right], \\ = \sum_{m,n=0}^{N-1} \frac{A_{mn}(1-z)^{-(2N+1)/2}}{(-it)^N\left(\partial F/\partial z\right)^N} &\left[\left(N+m-\frac{1}{2}\right) z^{-m}\left(\frac{\partial F}{\partial z}\right)^{-n}\right. \\ &\left. - m z^{-m-1}\left(\frac{\partial F}{\partial z}\right)^{-n} - (N+n) z^{-m-1}\left(\frac{\partial F}{\partial z}\right)^{-n-1}\right],\end{aligned}$$

which can be rearranged to an expression in the required form. Note that the only term with $n = N$ is the one with $m = n = N$, by the inductive hypothesis. ◻

Given the three lemmas above, we are now in a position to compute the large-t asymptotics of J_{B2}.

Lemma 3.2.12 (Large-t asymptotic expansion of J_{B2})**.** *We have*

$$J_{B2}(t;\lambda) = \sum_{j=1}^{N-1} T_j(t;\lambda) + R_N(t;\lambda), \tag{250}$$

where the terms T_j are defined by

$$T_j(t;\lambda) = -\left(\frac{-1}{it\frac{\partial F}{\partial z}} \cdot \frac{\partial}{\partial z}\right)^j \left(\frac{(1-z)^{-1/2}}{it\frac{\partial F}{\partial z}}\right) e^{itF(z;\lambda)}\Bigg|_{z=1-k} \tag{251}$$

and the remainder term R_N is defined by

$$R_N(t;\lambda) = \int_{1-k}^{\infty e^{i\phi}} \left(\frac{\partial}{\partial z} \cdot \frac{-1}{it\frac{\partial F}{\partial z}}\right)^N \left((1-z)^{-1/2}\right) e^{itF(z;\lambda)}\,\mathrm{d}z.$$

If ϕ is defined by (248) *and k satisfies*

$$(kt)^{\epsilon - \frac{m}{2m+1}} \ll D_- \ll 1 \tag{252}$$

for some $m \in \mathbb{N}$ and $\epsilon > 0$, then:

$$T_j(t;\lambda) = O\left((2j-1)!!t^{-j-1}k^{-(2j+1)/2}D_-^{-2j-1}\right); \tag{253}$$

$$R_N(t;\lambda) = O\left((2N-1)!!t^{-N}(\log t)^{(2N+1)/2}k^{-(2N-1)/2}D_-^{-2N}\right); \tag{254}$$

and thus (250) *provides a valid large-t asymptotic expansion.*

Proof. We apply integration by parts N times to the definition (238) of J_{B2} to obtain

$$J_{B2}(t;\lambda) = \sum_{j=0}^{N-1} \left[\left(\frac{-1}{it\frac{\partial F}{\partial z}} \cdot \frac{\partial}{\partial z}\right)^j \left(\frac{(1-z)^{-1/2}}{it\frac{\partial F}{\partial z}}\right) e^{itF(z;\lambda)}\right]_{1-k}^{\infty e^{i\phi}} + \int_{1-k}^{\infty e^{i\phi}} \left(\frac{\partial}{\partial z} \cdot \frac{-1}{it\frac{\partial F}{\partial z}}\right)^N \left((1-z)^{-1/2}\right) e^{itF(z;\lambda)}\,\mathrm{d}z, \tag{255}$$

for any $N \in \mathbb{N}$. By Lemmas 3.2.6 and 3.2.10 applied to the series expression given by Lemma 3.2.11, the $\infty e^{i\phi}$ parts of the boundary terms contribute nothing, and thus (255) becomes (250).

We now concentrate on proving the bounds (253) and (254). By Lemma 3.2.10, e^{itF} is bounded above by 1. Using this fact, along with Corollary 3.2.9 and Lemma 3.2.11, we find:

$$
\begin{aligned}
T_j(t;\lambda) &= \left[\frac{-(1-z)^{-(2j+1)/2}}{(-it)^{j+1}\left(\partial F/\partial z\right)^{j+1}}\sum_{m,n=0}^{j}\left[A_{mn}z^{-m}\left(\frac{\partial F}{\partial z}\right)^{-n}\right]e^{itF(z;\lambda)}\right]_{z=1-k}, \\
&= \frac{-k^{-(2j+1)/2}}{(-it)^{j+1}D^{j+1}}\sum_{m,n=0}^{j}\left[A_{mn}(1-k)^{-m}D^{-n}\right]e^{itF(1-k;\lambda)}, \\
&= O\left((2j-1)!!t^{-j-1}k^{-(2j+1)/2}\sum_{n=0}^{j}D_-^{-j-n-1}\right),
\end{aligned}
$$

which gives the required expression (253), since by (252) we are assuming that $D_- \ll 1$.

Using Lemma 3.2.11 again, we have

$$
R_N(t;\lambda) = \int_{1-k}^{\infty e^{i\phi}}\left[\frac{(1-z)^{-(2N+1)/2}}{(-it)^N\left(\partial F/\partial z\right)^N}\sum_{m,n=0}^{N}A_{mn}z^{-m}\left(\frac{\partial F}{\partial z}\right)^{-n}\right]e^{itF(z;\lambda)}\,\mathrm{d}z.
$$

From the definition (241) of the contour of integration, we have $|z| \geq 1-k$, and then, since $k = o(1)$ as $t \to \infty$ (from (239)), we have $|z| \geq \frac{1}{2}$, say, for t sufficiently large. Using this fact, along with Corollary 3.2.9, Lemma 3.2.10, and the estimates from Lemma 3.2.11, we have:

$$
\begin{aligned}
R_N(t;\lambda) &= O\left(\int_{1-k}^{\infty e^{i\phi}}\frac{(1-z)^{-(2N+1)/2}}{t^N}\sum_{m,n=0}^{N}\left|A_{mn}\right|\left|\frac{\partial F}{\partial z}\right|^{-N-n}\mathrm{d}z\right), \\
&= O\left(\frac{(2N-1)!!}{t^N}D_-^{-2N}\int_{1-k}^{\infty e^{i\phi}}\left|(1-z)^{-(2N+1)/2}\right|\mathrm{d}z\right).
\end{aligned}
$$

Here again we have implicitly used the assumption that $D_- \ll 1$ from (252). Now, from (244),

$$
|1-z|^2 = (k-R)^2\cos\phi + (k^2+R^2)(1-\cos\phi) \geq \frac{(k+R)^2}{2}(1-\cos\phi),
$$

where we have used the inequality $(k+R)^2 \leq 2(k^2+R^2)$. Therefore,

$$
\frac{1}{|1-z|^{(2N+1)/2}} \leq \left(\frac{2}{1-\cos\phi}\right)^{(2N+1)/4}\frac{1}{(k+R)^{(2N+1)/2}}.
$$

For $\log\lambda \geq 0$, the first equation in (248) implies that $(1-\cos\phi)^{-1} = O(1)$. For $\log\lambda < 0$, the second equation in (248) implies that

$$
\tan 2\phi = \frac{\pi}{|\log\lambda|}, \quad \cos 2\phi = \left(1+\tan^2 2\phi\right)^{-1/2} = \left(1+\frac{\pi^2}{|\log\lambda|^2}\right)^{-1/2},
$$

and

$$(1-\cos\phi)^{-1} = \left(1-\sqrt{\frac{1}{2}\left(1+\left(1+\frac{\pi^2}{|\log\lambda|^2}\right)^{-1/2}\right)}\right)^{-1}. \tag{256}$$

Therefore, as $|\log\lambda|$ increases, $\tan 2\phi$ decreases, $\cos 2\phi$ increases, and $(1-\cos\phi)^{-1}$ increases. Thus, the maximal value of $(1-\cos\phi)^{-1}$ is achieved when λ is minimal, i.e. when $\lambda = (t^{1-\delta}-1)^{-1}$. Substituting this into (256) we find that

$$(1-\cos\phi)^{-1} \sim \frac{8|\log\lambda|^2}{\pi^2} = O\big((\log t)^2\big).$$

Therefore, in general we have

$$|1-z|^{-(2N+1)/2} = O\left((\log t)^{(2N+1)/2}(k+R)^{-(2N+1)/2}\right),$$

and then

$$\begin{aligned} R_N(t;\lambda) &= O\left(\frac{(2N-1)!!}{t^N}D_-^{-2N}\int_0^\infty (\log t)^{(2N+1)/2}(k+R)^{-(2N+1)/2}\,\mathrm{d}R\right), \\ &= O\left((2N-1)!!t^{-N}(\log t)^{(2N+1)/2}D_-^{-2N}\Big[(k+R)^{-(2N-1)/2}\Big]_{R=0}^{\infty}\right), \\ &= O\left((2N-1)!!t^{-N}(\log t)^{(2N+1)/2}D_-^{-2N}k^{-(2N-1)/2}\right), \end{aligned}$$

as required.

Finally, it remains to check that (250) with the estimates (253) and (254) actually gives a valid asymptotic formula, i.e. that each term in the series is smaller than the next term for sufficiently large t. From (253) we see that the estimate for T_{j+1} is smaller than the one for T_j if and only if

$$t^{-1}k^{-1}D_-^{-2} \ll 1,$$

i.e. if and only if $D_- \gg (kt)^{-1/2}$, which is true by the first half of (252). Furthermore, from (253) and (254) we see that the estimate for R_N is smaller than the one for *some* T_j (not necessarily T_{N-1}) if and only if

$$t^{-N+j+1}(\log t)^{(2N+1)/2}k^{-N+j+1}D_-^{-2N+2j+1} \ll 1,$$

which is equivalent to the first half of (252) with $m = N-j-1$, since we know k behaves like a power of t by the second half of (252). □

Remark 3.2.13. We just saw that our estimate for R_N is smaller than the one for T_j iff the second half of (252) holds with m replaced by $N-j-1$. Given that this is true, the same statement must hold for any smaller value of j, corresponding to larger values of $N-j-1$. (If $D_- \gg (kt)^{\epsilon-\frac{m}{2m+1}}$ holds for some m, then it also holds for any larger value of m, since $kt \gg 1$ by (252).)

Thus we have shown that, if m is minimal for (252) to be valid, then our estimate for R_N is smaller than the one for T_j if $j \leq N-m-1$, and larger than the one for T_j if $j > N-m$. There is a clear cut-off point beyond which the terms of the asymptotic series no longer necessarily dominate the remainder term.

3.2.4 Step 3: The asymptotics of J_{B1}

Lemma 3.2.14 (Large-t asymptotic expansion of J_{B1})**.** *Let $a := 1 - k\,t^{1-\delta}$, so that*

$$k = t^{\delta-1}(1-a), \tag{257}$$

and assume that this new variable a satisfies

$$t^{-\delta/2} \ll a \ll t^{-\delta/3}. \tag{258}$$

Then, the large-t asymptotic expansion of J_{B1} is given to first order by

$$J_{B1}(t;\lambda) = \exp\left(itF(1-t^{\delta-1};\lambda) - i\omega^2\right) t^{-1/2}\sqrt{\frac{2}{1+\lambda_c}}\left(\int_\omega^{\omega+a\sqrt{\lambda_c t/2}} e^{i\xi^2}\,\mathrm{d}\xi\right) + O\left(t^{-\frac{1}{2}+\frac{3\delta}{2}}a^4\right), \tag{259}$$

where ω is defined by (229)*.*

Proof. We start by making two changes of variable in the expression (237) for J_{B1}, in order to improve the notation. Firstly, substituting $x = 1-z$ yields the simpler expression

$$J_{B1}(t;\lambda) = \int_k^{t^{\delta-1}} x^{-1/2}\exp\left(itF(1-x,\lambda)\right)\mathrm{d}x.$$

Then, in order to have the critical value at 0, we substitute $x = t^{\delta-1}(1-\zeta)$. Thus $\mathrm{d}x/\mathrm{d}\zeta = -t^{\delta-1}$, and the value $x = t^{\delta-1}$ corresponds to $\zeta = 0$ as desired, while the value $x = k$ corresponds to $\zeta = 1 - kt^{1-\delta} < 1$. We get:

$$J_{B1}(t;\lambda) = t^{(\delta-1)/2}\int_0^{1-kt^{1-\delta}} (1-\zeta)^{-1/2}\exp\left(itF\left(1-t^{\delta-1}(1-\zeta);\lambda\right)\right)\mathrm{d}\zeta. \tag{260}$$

We now expand the exponent in powers of ζ and then show that the higher powers can be discarded without affecting the leading-order asymptotics of J_{B1}. Indeed, expanding the exponent, and using the definition (227) of λ_c, we have

$$
\begin{aligned}
F(1-t^{\delta-1}(1-\zeta),\lambda) &= \big(1-t^{\delta-1}(1-\zeta)\big)\log\big(1-t^{\delta-1}(1-\zeta)\big)\\
&\qquad + t^{\delta-1}(1-\zeta)\log\big(t^{\delta-1}(1-\zeta)\big) + \big(1-t^{\delta-1}(1-\zeta)\big)\log\lambda\\
&= \big(1-t^{\delta-1}+t^{\delta-1}\zeta\big)\log\big(1-t^{\delta-1}+t^{\delta-1}\zeta\big)\\
&\qquad + t^{\delta-1}(1-\zeta)\log\big(t^{\delta-1}(1-\zeta)\big) + \big(1-t^{\delta-1}+t^{\delta-1}\zeta\big)\log\lambda\\
&= \big(1-t^{\delta-1}+t^{\delta-1}\zeta\big)\Big[\log\big(1-t^{\delta-1}\big)+\log(1+\lambda_c\zeta)\Big]\\
&\qquad + t^{\delta-1}(1-\zeta)\Big[\log\big(t^{\delta-1}\big)+\log(1-\zeta)\Big] + \big(1-t^{\delta-1}+t^{\delta-1}\zeta\big)\log\lambda\\
&= \big(1-t^{\delta-1}+t^{\delta-1}\zeta\big)\Big[\log\big(1-t^{\delta-1}\big)+\lambda_c\zeta-\frac{\lambda_c^2\zeta^2}{2}+\frac{\lambda_c^3\zeta^3}{3}+\dots\Big]\\
&\qquad + t^{\delta-1}(1-\zeta)\Big[\log\big(t^{\delta-1}\big)-\zeta-\frac{\zeta^2}{2}-\frac{\zeta^3}{3}-\dots)\Big]\\
&\qquad + \big(1-t^{\delta-1}+t^{\delta-1}\zeta\big)\log\lambda\\
&= c_0+c_1\zeta+c_2\zeta^2+c_3\zeta^3+\dots,
\end{aligned}
\tag{261}
$$

where the coefficients c_j are evaluated as follows:

$$
\begin{aligned}
c_0 &= \big(1-t^{\delta-1}\big)\log\big(1-t^{\delta-1}\big)+t^{\delta-1}\log t^{\delta-1}+\big(1-t^{\delta-1}\big)\log\lambda = F\big(1-t^{\delta-1};\lambda\big);\\
c_1 &= \big(1-t^{\delta-1}\big)\lambda_c+t^{\delta-1}\log\big(1-t^{\delta-1}\big)-t^{\delta-1}-t^{\delta-1}\log t^{\delta-1}+t^{\delta-1}\log\lambda = t^{\delta-1}\log\left(\frac{\lambda}{\lambda_c}\right);\\
c_n &= \big(1-t^{\delta-1}\big)\Big[\frac{(-1)^{n+1}\lambda_c^n}{n}\Big]+t^{\delta-1}\Big[\frac{(-1)^n\lambda_c^{n-1}}{n-1}\Big]+t^{\delta-1}\left(\frac{-1}{n}\right)-t^{\delta-1}\left(\frac{-1}{n-1}\right)\\
&= (-1)^n\Big[\frac{-t^{\delta-1}\lambda_c^{n-1}}{n}+\frac{t^{\delta-1}\lambda_c^{n-1}}{n-1}\Big]+\frac{t^{\delta-1}}{n(n-1)}\\
&= \frac{t^{\delta-1}}{n(n-1)}\big(1-(-\lambda_c)^{n-1}\big)\sim\frac{t^{\delta-1}}{n(n-1)}\ \text{for all } n\ge 2.
\end{aligned}
$$

Using (261) in (260), we find:

$$
\begin{aligned}
J_{B1}(t;\lambda) &= t^{(\delta-1)/2}\int_0^{1-kt^{1-\delta}}(1-\zeta)^{-1/2}\,e^{it(c_0+c_1\zeta+c_2\zeta^2)}\,e^{it(c_3\zeta^3+c_4\zeta^4+\dots)}\,\mathrm{d}\zeta,\\
&= e^{itc_0}t^{(\delta-1)/2}\int_0^{1-kt^{1-\delta}}(1-\zeta)^{-1/2}\,e^{it(c_1\zeta+c_2\zeta^2)}\,\mathrm{d}\zeta+I_R(t;\lambda),
\end{aligned}
\tag{262}
$$

where the remainder term I_R is given by

$$
\begin{aligned}
I_R(t;\lambda) &= t^{(\delta-1)/2}\int_0^{1-kt^{1-\delta}}(1-\zeta)^{-1/2}e^{it(c_0+c_1\zeta+c_2\zeta^2)}\left(e^{it(c_3\zeta^3+c_4\zeta^4+\dots)}-1\right)\mathrm{d}\zeta,\\
&= t^{(\delta-1)/2}\int_0^{a}(1-\zeta)^{-1/2}O\left(e^{it(c_3\zeta^3+c_4\zeta^4+\dots)}-1\right)\mathrm{d}\zeta.
\end{aligned}
$$

The motivation for the substitution (257) now becomes clear: it simplifies the upper bound of the integral from $1-kt^{1-\delta}$ to simply a. To obtain the required result (259), we will first prove that, under the assumption (258), we have $I_R = O\big(t^{-\frac{1}{2}+\frac{3\delta}{2}}a^4\big)$.

Firstly, the exponent appearing in the integrand of I_R is:

$$it\sum_{n=3}^{\infty}c_n\zeta^n = it^\delta\sum_{n=3}^{\infty}\frac{\zeta^n}{n(n-1)}\left(1-(-\lambda_c)^{n-1}\right) = it^\delta\left(\frac{\zeta^3}{6}+O\left(\sum_{n=4}^{\infty}\frac{\zeta^n}{12}\left(1+\lambda_c^3\right)\right)\right),$$
$$= it^\delta\left(\frac{\zeta^3}{6}+O\left(\frac{\zeta^4}{6}(1-\zeta)^{-1}\right)\right) = \frac{it^\delta\zeta^3}{6}+O\left(t^\delta\zeta^4\right).$$

So I_R itself can be estimated as follows:

$$I_R(t;\lambda) = t^{(\delta-1)/2}\int_0^a(1-\zeta)^{-1/2}O\left(\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{it^\delta\zeta^3}{6}+O\left(t^\delta\zeta^4\right)\right)^n\right)\mathrm{d}\zeta,$$
$$= O\left(t^{(\delta-1)/2}\int_0^a\left(\frac{it^\delta\zeta^3}{6}+O\left(t^\delta\zeta^4,t^{2\delta}\zeta^6\right)\right)\mathrm{d}\zeta\right)\ \text{as } t\to\infty,$$

where we have used the second half of (258), or equivalently $t^\delta a^3 \ll 1$, to ensure that the powers of $t^\delta\zeta^3$ in the exponential expansion do not increase to infinity. The second half of (258) also implies that $a \ll 1$, and so all higher-order terms, whether of the form $t^\delta\zeta^{3+K}$ or $\left(t^\delta\zeta^3\right)^K$ or a combination of both, are negligible compared to the leading term $t^\delta\zeta^3$. Thus I_R satisfies

$$I_R(t;\lambda) = O\left(t^{(\delta-1)/2}\int_0^a\left(\frac{it^\delta\zeta^3}{6}\right)\mathrm{d}\zeta\right) = O\left(t^{-\frac{1}{2}+\frac{3\delta}{2}}a^4\right).$$

Hence, equation (262) becomes:

$$J_{B1}(t;\lambda) = e^{itc_0}t^{(\delta-1)/2}\int_0^a(1-\zeta)^{-1/2}e^{it(c_1\zeta+c_2\zeta^2)}\,\mathrm{d}\zeta + O\left(t^{-\frac{1}{2}+\frac{3\delta}{2}}a^4\right),$$
$$= e^{itc_0}t^{(\delta-1)/2}\int_0^a\left(1+O(\zeta)\right)e^{it(c_1\zeta+c_2\zeta^2)}\,\mathrm{d}\zeta + O\left(t^{-\frac{1}{2}+\frac{3\delta}{2}}a^4\right),$$
$$= e^{itc_0}t^{(\delta-1)/2}\int_0^a e^{it(c_1\zeta+c_2\zeta^2)}\,\mathrm{d}\zeta + O\left(t^{(\delta-1)/2}\int_0^a\zeta\mathrm{d}\zeta\right) + O\left(t^{-\frac{1}{2}+\frac{3\delta}{2}}a^4\right),$$
$$= e^{itc_0}t^{(\delta-1)/2}\int_0^a e^{it(c_1\zeta+c_2\zeta^2)}\,\mathrm{d}\zeta + O\left(t^{-\frac{1}{2}+\frac{\delta}{2}}a^2\right) + O\left(t^{-\frac{1}{2}+\frac{3\delta}{2}}a^4\right).$$

Since the first half of (258) gives $t^{-\frac{1}{2}+\frac{\delta}{2}}a^2 \ll t^{-\frac{1}{2}+\frac{3\delta}{2}}a^4$, we obtain

$$J_{B1}(t;\lambda) = e^{itF(1-t^{\delta-1};\lambda)}t^{(\delta-1)/2}\int_0^a\exp\left[it^\delta\left((\log\tfrac{\lambda}{\lambda_c})\zeta+\tfrac{1}{2}(1+\lambda_c)\zeta^2\right)\right]\mathrm{d}\zeta + O\left(t^{-\frac{1}{2}+\frac{3\delta}{2}}a^4\right). \tag{263}$$

We now manipulate the integral on the right-hand side of (263) to obtain the desired result (259). Using the fact that

$$t\lambda_c = t^\delta(1+\lambda_c) \tag{264}$$

(which follows from the definition (227) of λ_c), we find that the exponent of the integrand in (263) is

$$\begin{aligned} it^\delta\Bigg(\log\left(\frac{\lambda}{\lambda_c}\right)\zeta + \frac{1}{2}(1+\lambda_c)\zeta^2\Bigg) &= \frac{it^\delta(1+\lambda_c)}{2}\left(\zeta^2 + 2\frac{\log(1+\Lambda)}{1+\lambda_c}\zeta\right), \\ &= \frac{it\lambda_c}{2}\left(\zeta^2 + 2\sqrt{\frac{2}{\lambda t}}\omega\zeta\right), \\ &= \frac{it\lambda_c}{2}\left(\zeta + \sqrt{\frac{2}{\lambda_c t}}\omega\right)^2 - i\omega^2. \end{aligned}$$

So using the change of variables $\xi = \omega + \zeta\sqrt{t\lambda_c/2}$, the integral term in (263) can be written as

$$\begin{aligned} &\exp\left(itF(1-t^{\delta-1};\lambda) - i\omega^2\right)t^{(\delta-1)/2}\int_0^a \exp\left(\frac{it\lambda_c}{2}\left(\zeta + \sqrt{\frac{2}{\lambda_c t}}\omega\right)^2\right)\mathrm{d}\zeta \\ &= \exp\left(itF(1-t^{\delta-1};\lambda) - i\omega^2\right)t^{(\delta-1)/2}\sqrt{\frac{2}{\lambda_c t}}\int_\omega^{\omega+a\sqrt{\lambda_c t/2}} e^{i\xi^2}\,\mathrm{d}\xi, \end{aligned}$$

which becomes the main term in (259) after using (264). □

3.2.5 Step 4: Combining and unifying the asymptotics

We now prove Theorem 3.2.5 by combining the results of Lemmas 3.2.12 and 3.2.14. This is where we need to be very precise about our choice of the splitting point k, or equivalently of the variable a defined by (257), in order for these two results to be compatible.

Proof of Theorem 3.2.5. When deriving the asymptotics for J_{B2} in Lemma 3.2.12, we were still using a fairly general parameter k, required only to satisfy the condition (252). But in Lemma 3.2.14 we used a much more specific form of k, namely that given by (257) with a satisfying (258). In order to combine the asymptotics of J_{B1} and J_{B2}, we first rewrite the results of Lemma 3.2.12 with k replaced by $t^{\delta-1}(1-a)$.

Firstly, we have

$$D_- = \log\left(\frac{t^{\delta-1}}{t^{\delta-1}(1-a)}\right) = -\log(1-a) \sim a,$$

and so the assumption (252) can be rewritten as

$$t^{\epsilon\delta-\frac{m\delta}{2m+1}} \ll a \ll 1. \tag{265}$$

Note that taking $m = 1$ would make (258) and (265) contradict each other, and so we must have

$$t^{\epsilon\delta-\frac{m\delta}{2m+1}} \ll a \ll t^{-\delta/3}$$

for some $m \geq 2$ and $\epsilon > 0$. By Lemma 3.2.12, we then find that the estimate (250) holds with

$$T_j(t;\lambda) = O\left((2j-1)!!t^{-\frac{1}{2}-\frac{(2j+1)\delta}{2}}a^{-2j-1}\right) \tag{266}$$

for each j, and

$$R_N(t;\lambda) = O\left((2N-1)!!t^{-\frac{1}{2}-\frac{(2N-1)\delta}{2}}(\log t)^{(2N+1)/2}a^{-2N}\right).$$

We assume that m is minimal for the assumption (265) to be valid, i.e. that

$$t^{\epsilon\delta-\frac{m\delta}{2m+1}} \ll a \ll t^{\epsilon\delta-\frac{(m-1)\delta}{2m-1}},$$

which is implied by (236). By Remark 3.2.13, this means our estimate for R_N is smaller than the one for T_{N-m-1} but larger than the one for T_{N-m}. Thus the estimate (250) becomes:

$$\begin{aligned} J_{B2}(t;\lambda) = & \sum_{j=1}^{N-m-1} T_j(t;\lambda) + \sum_{j=N-m}^{N-1} O\left((2j-1)!!t^{-\frac{1}{2}-\frac{(2j+1)\delta}{2}}a^{-2j-1}\right) \\ & \qquad + O\left((2N-1)!!t^{-\frac{1}{2}-\frac{(2N-1)\delta}{2}}(\log t)^{(2N+1)/2}a^{-2N}\right), \\ = & \sum_{j=1}^{N-m-1} T_j(t;\lambda) + O\left((2N-1)!!t^{-\frac{1}{2}-\frac{(2N-1)\delta}{2}}(\log t)^{(2N+1)/2}a^{-2N}\right), \end{aligned} \tag{267}$$

where the error term is sufficiently small compared to the rest that it doesn't swallow up any of the remaining series.

Since we need to combine this result with the asymptotic formula (259) for J_{B1}, we would like to ensure that the error term of J_{B1}, namely $O\big(t^{-\frac{1}{2}+\frac{3\delta}{2}}a^4\big)$, is *also* sufficiently small so that it doesn't swallow up any of the terms in the above series. In other words, we require that our estimate for T_j should be $\gg t^{-\frac{1}{2}+\frac{3\delta}{2}}a^4$ for all $j \leq N-m-1$. Checking this condition, we obtain

$$t^{-\frac{1}{2}-\frac{(2j+1)\delta}{2}}a^{-2j-1} \gg t^{-\frac{1}{2}+\frac{3\delta}{2}}a^4, \quad \text{which holds iff} \quad a \ll t^{-(j+2)\delta/(2j+5)},$$

which, by the assumption (236), is true provided that $j+2 \leq m-1$. So we set $N-m-1 =$

$m-3$, i.e. $N = 2m-2$. Now combining the two asymptotic expansions (259) and (267) gives the required result (235). $\square$

Remark 3.2.15 (Comparison of error terms). Note that as the order m of the asymptotics increases, the parameter a decreases (assuming (236)), so the error term from J_{B1} becomes smaller and smaller in comparison to the series from J_{B2}. This makes sense, because when a is very small, k is very close to $t^{\delta-1}$, i.e. the integral in J_{B2} is closer to the stationary point while the one in J_{B1} is shorter, and so J_{B1} contributes less to the final answer.

More rigorously comparing the error terms in (267) and (259), we find that, with our choice of $N = 2m-2$,

$$t^{-\frac{1}{2}-\frac{(2N-1)\delta}{2}}(\log t)^{(2N+1)/2}a^{-2N} \ll t^{-\frac{1}{2}+\frac{3\delta}{2}}a^4 \quad \text{iff} \quad a \gg t^{-\frac{\delta}{2}+\frac{\delta}{4m}}(\log t)^{\frac{1}{2}-\frac{3}{8m}}. \tag{268}$$

If we assume a takes the form $a = t^{-b\delta}$ for some constant b, then we can ignore the log terms. This is because the condition (236) can be rewritten as

$$\frac{1}{2}-\frac{1}{4m-2} < b < \frac{1}{2}-\frac{1}{4m+2},$$

and the cutoff point for which error term is dominant, given by (268), is

$$b = \frac{1}{2}-\frac{1}{4m}, \tag{269}$$

which is, in some sense, directly in the middle of the interval of possible values for b.

Corollary 3.2.16. *The leading-order asymptotics of J_B are given by*

$$\begin{aligned} J_B(t;\lambda) &= i\,\exp\left(itF(1-k;\lambda)\right)t^{-\frac{1}{2}-\frac{\delta}{2}}\left[\log\left(\frac{1}{k}-1\right)+\log\lambda\right]^{-1} \\ &\quad + \exp\left(itF(1-t^{\delta-1};\lambda)-i\omega^2\right)t^{-1/2}\sqrt{\frac{2}{1+\lambda_c}}\left(\int_{\omega}^{\omega+a\sqrt{\lambda_c t/2}} e^{i\xi^2}\,\mathrm{d}\xi\right)+O\left(t^{\epsilon-\frac{1}{2}-\frac{\delta}{4}}\right), \end{aligned} \tag{270}$$

for any $\epsilon > 0$, where $k = t^{\delta-1}(1-a)$, $a = t^{-7\delta/16}$, and ω is defined by (229)*.*

Moreover, (270) *agrees with the asymptotics of J_B found in Theorem 3.2.3 in the case $\sigma = 1/2$.*

Proof. We take $m = 4$, the lowest possible value of m, and let $a = t^{-b\delta}$ as in Remark 3.2.15. By (269), the value of a required to make both error terms in (235) of comparable size is given by

$$a = t^{-\frac{\delta}{2}+\frac{\delta}{4m}} = t^{-7\delta/16}.$$

With this choice of a, the two error terms in (235) are

$$O\left(t^{-\frac{1}{2}-\frac{\delta}{4}}(\log t)^{(13/2)}\right) \quad \text{and} \quad O\left(t^{-\frac{1}{2}-\frac{\delta}{4}}\right),$$

which are both $O\big(t^{\epsilon-\frac{1}{2}-\frac{\delta}{4}}\big)$ as required.

Also, from the definition (251) of T_j, we have

$$T_1(t;\lambda) = \left[\left(\frac{(1-z)^{-1/2}}{it\frac{\partial F}{\partial z}}\right)e^{itF(z;\lambda)}\right]_{z=1-k}^{\infty e^{i\phi}} = \frac{-k^{-1/2}e^{itF(1-k;\lambda)}}{it\frac{\partial F}{\partial z}\Big|_{z=1-k}} = ie^{itF(1-k;\lambda)}t^{-1}k^{-1/2}D^{-1}.$$

Then, using the definition (257) of a and the definition (247) of D, we find

$$\begin{aligned} T_1(t;\lambda) &= ie^{itF(1-k;\lambda)}t^{-\frac{1}{2}-\frac{\delta}{2}}(1-a)^{-1/2}D^{-1} = ie^{itF(1-k;\lambda)}t^{-\frac{1}{2}-\frac{\delta}{2}}D^{-1} + O\left(t^{-\frac{1}{2}-\frac{\delta}{2}}aD_-^{-1}\right), \\ &= ie^{itF(1-k;\lambda)}t^{-\frac{1}{2}-\frac{\delta}{2}}\left[\log\left(\frac{1}{k}-1\right)+\log\lambda\right]^{-1} + O\left(t^{-\frac{1}{2}-\frac{\delta}{2}}\right), \end{aligned}$$

and this $O\big(t^{-\frac{1}{2}-\frac{\delta}{2}}\big)$ error term is absorbed by the $O\big(t^{-\frac{1}{2}-\frac{\delta}{4}}\big)$ error term in (235).

It remains to show that (270) agrees with the asymptotics in Theorem 3.2.3 in the case $\sigma = 1/2$. From (230) and (232) we have

$$J_B(t;\lambda) = \sqrt{\frac{2}{(1+\lambda_c)t}}\exp\left(\frac{itf_0(\Lambda,\lambda_c)}{1+\lambda_c} - i\omega^2\right)\left(\int_\omega^{\infty e^{i\pi/4}} e^{i\xi^2}\,\mathrm{d}\xi\right)\Big(1+o(1)\Big). \quad (271)$$

Using the definitions of f_0 (231) and λ_c (227), and some algebraic manipulation we find that

$$\frac{itf_0(\Lambda,\lambda_c)}{1+\lambda_c} - i\omega^2 = itF(1-t^{\delta-1};\lambda),$$

so (271) becomes

$$J_B(t;\lambda) = \exp\Big(itF(1-t^{\delta-1};\lambda)-i\omega^2\Big)t^{-1/2}\sqrt{\frac{2}{1+\lambda_c}}\left(\int_\omega^{\infty e^{i\pi/4}} e^{i\xi^2}\,\mathrm{d}\xi\right)\Big(1+o(1)\Big). \quad (272)$$

From (270) we have

$$J_B(t;\lambda) = \exp\Big(itF(1-t^{\delta-1};\lambda)-i\omega^2\Big)t^{-1/2}\sqrt{\frac{2}{1+\lambda_c}}\left(\int_\omega^{\infty e^{i\pi/4}} e^{i\xi^2}\,\mathrm{d}\xi\right) + r(t;\lambda), \quad (273)$$

where the remainder $r(t;\lambda)$ is defined by

$$r(t;\lambda) \coloneqq i\,\exp\big(itF(1-k;\lambda)\big)\,t^{-\frac{1}{2}-\frac{\delta}{2}}\left[\log\left(\frac{1}{k}-1\right)+\log\lambda\right]^{-1}$$
$$-\exp\Big(itF(1-t^{\delta-1};\lambda)-i\omega^2\Big)t^{-1/2}\sqrt{\frac{2}{1+\lambda_c}}\left(\int_{\omega+a\sqrt{\lambda_c t/2}}^{\infty e^{i\pi/4}} e^{i\xi^2}\,\mathrm{d}\xi\right). \tag{274}$$

If we can show that $r(t;\lambda)$ is little-o of the first term in (273) as $t\to\infty$ (independently of λ), then the asymptotics (273) obtained from Theorem 3.2.5 are the same as those (272) obtained from Theorem 3.2.3, and the proof is complete. The asymptotics of the first term in (273) (which we can obtain from (234) in Theorem 3.2.3) imply that it is sufficient to show that

$$r(t;\lambda)=\begin{cases} o(t^{-1/2}) & \text{when } \omega=O(1),\\ o\big(t^{-1/2}t^{-\delta/2}(\log t)^{-1}\big) & \text{when } \omega\to\infty.\end{cases} \tag{275}$$

Now, from the definitions of k (257), Λ (228), and λ_c (227), and the Taylor series for $\log(1+x)$,

$$\begin{aligned}\log\left(\frac{1}{k}-1\right)+\log\lambda &= \log\left(\left(\frac{t^{1-\delta}}{1-a}-1\right)\lambda_c(1+\Lambda)\right),\\ &= \log(1+\Lambda)+\log\left(\frac{1}{1-a}\right)+\log\left(1+\frac{a}{t^{1-\delta}-1}\right),\\ &= \log(1+\Lambda)+a+O(a^2)+\frac{a}{t^{1-\delta}}\left(1+O\left(\frac{1}{t^{1-\delta}}\right)\right) \qquad \text{as } t\to\infty.\end{aligned}$$

Using these asymptotics in the first term of (274), and using the first-order asymptotics $\int_z^{\infty e^{i\pi/4}} e^{i\xi^2}\,\mathrm{d}\xi = e^{iz^2}\left(\frac{-1}{2iz}+O(z^{-3})\right)$ (which follows from integration by parts) in the second term, we find that

$$\begin{aligned} r(t;\lambda) =& \frac{i\,\exp\big(itF(1-k;\lambda)\big)}{t^{1/2}t^{\delta/2}\Big[\log(1+\Lambda)+a+O(a^2)\Big]}\\ &-\frac{i\exp\Big(itF(1-t^{\delta-1};\lambda)-i\omega^2\Big)e^{i(\omega+a\sqrt{\lambda_c t/2})^2}}{t^{1/2}(1+\lambda_c)^{1/2}\sqrt{2}\left(\omega+a\sqrt{\frac{\lambda_c t}{2}}\right)}\left(1+O\left(\left(\omega+a\sqrt{\frac{\lambda_c t}{2}}\right)^{-2}\right)\right),\\ =& \frac{i\exp\big(itF(1-t^{\delta-1};\lambda)\big)}{t^{1/2}t^{\delta/2}}\Bigg(\frac{\exp\big(it\big(F(1-k;\lambda)-F(1-t^{\delta-1};\lambda)\big)\big)}{\log(1+\Lambda)+a+O(a^2)}\\ &-\frac{\exp\big(2i\omega a\sqrt{\lambda_c t/2}+ia^2\lambda_c t/2\big)}{\log(1+\Lambda)+a(1+\lambda_c)}\big(1+O(t^{-\delta/8})\big)\Bigg), \end{aligned} \tag{276}$$

where we have used both the definition of ω (229) and the equation (264).

We now use (229) and (264) to manipulate the final exponent in (276):

$$2\omega a\sqrt{\frac{\lambda_c t}{2}} + \frac{1}{2}a^2\lambda_c t = t^\delta a \log\left(\frac{\lambda}{\lambda_c}\right) + \frac{1}{2}a^2 t^\delta (1+\lambda_c).$$

Therefore, the required asymptotics of $r(t;\lambda)$ (275) will follow from (276) if we can show that

$$t\Big(F(1-k;\lambda) - F(1-t^{\delta-1};\lambda)\Big) = t^\delta a \log\left(\frac{\lambda}{\lambda_c}\right) + \frac{1}{2}a^2 t^\delta(1+\lambda_c) + O(t^{-\epsilon}) \tag{277}$$

for some $\epsilon > 0$. The definition of $F(z;\lambda)$ (224) implies that

$$F(z;\lambda) - F(w;\lambda) = (z-w)\left[\log\left(\frac{w}{1-w}\right) + \log\lambda\right] + (1-z)\log\left(\frac{1-z}{1-w}\right) + z\log\left(\frac{z}{w}\right).$$

Using this, along with some algebraic manipulation, the equation (264), and the Taylor series for $\log(1+x)$, we find that the left-hand side of (277) equals

$$\begin{aligned} t^\delta a \log\left(\frac{\lambda}{\lambda_c}\right) &+ t^\delta(1-a)\log(1-a) + \big(t - t^\delta(1-a)\big)\log(1+a\lambda_c) \\ &= t^\delta a \log\left(\frac{\lambda}{\lambda_c}\right) + \frac{1}{2}a^2 t^\delta(1+\lambda_c) + O(a^3 t^\delta), \end{aligned}$$

which is the right-hand side of (277), since $a^3 t^\delta = t^{-5\delta/16}$; the proof is therefore complete. □

Remark 3.2.17. The problem considered in this chapter has already been justified as an object of interest, due to its involvement in the work [56] towards the Lindelöf Hypothesis. However, it may also have some significance with regard to the previous chapter.

Our Assumption 3.1.9, which we required in order to achieve valid asymptotics of the Hurwitz zeta function to all orders, was introduced for the precise purpose of avoiding integrals coming too close to stationary points. Recall that we used integration by parts arguments for every section of the asymptotic analysis in §3.1, and the ϵ assumption on η was introduced so that we could remain uniformly far from the stationary point in each case and use only integration by parts.

However, thanks to the work of this chapter, we have now seen a way of establishing uniform asymptotics in the neighbourhood of a stationary point. If the same methodology can be applied to the integrals involved in §3.1, then we may be able to eliminate Assumption 3.1.9 altogether, and obtain a uniform analysis for the asymptotics to all orders of the Hurwitz zeta function, regardless of the positioning of the stationary points relative to the curve.

Part 4

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